

MOMENTUM RAY TRANSFORMS

VENKATESWARAN P. KRISHNAN^{*†}, RAMESH MANNA[†], SUMAN KUMAR SAHOO[†] AND
VLADIMIR A. SHARAFUTDINOV[#]

ABSTRACT. The momentum ray transform I^k integrates a rank m symmetric tensor field f over lines with the weight t^k : $(I^k f)(x, \xi) = \int_{-\infty}^{\infty} t^k \langle f(x + t\xi), \xi^m \rangle dt$. In particular, the ray transform $I = I^0$ was studied by several authors since it had many tomographic applications. We present an algorithm for recovering f from the data $(I^0 f, I^1 f, \dots, I^m f)$. In the cases of $m = 1$ and $m = 2$, we derive the Reshetnyak formula that expresses $\|f\|_{H_t^s(\mathbb{R}^n)}$ through some norm of $(I^0 f, I^1 f, \dots, I^m f)$. The H_t^s -norm is a modification of the Sobolev norm weighted differently at high and low frequencies. Using the Reshetnyak formula, we obtain a stability estimate.

1. INTRODUCTION

The ray transform integrates symmetric tensor fields over straight lines. Let $\langle \cdot, \cdot \rangle$ be the standard dot-product on \mathbb{R}^n and $|\cdot|$, the corresponding norm. The family of oriented straight lines in \mathbb{R}^n is parameterized by points of the manifold

$$T\mathbb{S}^{n-1} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\xi| = 1, \langle x, \xi \rangle = 0\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

that is the tangent bundle of the unit sphere \mathbb{S}^{n-1} . Namely, a point $(x, \xi) \in T\mathbb{S}^{n-1}$ determines the line $\{x + t\xi \mid t \in \mathbb{R}\}$. Along with the space $C^\infty(T\mathbb{S}^{n-1})$ of smooth functions, we use the Schwartz space $\mathcal{S}(T\mathbb{S}^{n-1})$. Observe that the space $\mathcal{S}(E)$ is well defined for a smooth vector bundle $E \rightarrow M$ over a compact manifold M .

Let $S^m \mathbb{R}^n$ be the complex vector space of rank m symmetric tensors on \mathbb{R}^n . The dimension of $S^m \mathbb{R}^n$ is $\binom{n+m-1}{m}$. In particular, $S^0 \mathbb{R}^n = \mathbb{C}$ and $S^1 \mathbb{R}^n = \mathbb{C}^n$. Let $\mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$ be the Schwartz space of $S^m \mathbb{R}^n$ -valued functions that are called *rank m smooth fast decaying symmetric tensor fields* on \mathbb{R}^n . The *ray transform* is the linear bounded operator

$$I : \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n) \rightarrow \mathcal{S}(T\mathbb{S}^{n-1}) \tag{1.1}$$

that is defined, for $f = (f_{i_1 \dots i_m}) \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$, by

$$If(x, \xi) = \int_{-\infty}^{\infty} f_{i_1 \dots i_m}(x + t\xi) \xi^{i_1} \dots \xi^{i_m} dt = \int_{-\infty}^{\infty} \langle f(x + t\xi), \xi^m \rangle dt \quad ((x, \xi) \in T\mathbb{S}^{n-1}). \tag{1.2}$$

We use the Einstein summation rule: the summation from 1 to n is assumed over every index repeated in lower and upper positions in a monomial. To adopt our formulas to the Einstein summation rule, we use either lower or upper indices for denoting coordinates of

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vectors and tensors. For instance, $\xi^i = \xi_i$ in (1.2). There is no difference between covariant and contravariant tensors since we use Cartesian coordinates only. Being initially defined by (1.2) on smooth fast decaying tensor fields, the operator (1.1) then extends to some wider spaces of tensor fields.

The study of the ray transform is motivated by several applications. In the case of $m = 0$ (when f is a function), the ray transform is the main mathematical tool of Computer Tomography. In the case of $m = 1$ (when f is a vector field), the operator I is called the *Doppler transform* and serves as the main mathematical tool of Doppler Tomography. In the cases of $m = 2$ and of $m = 4$, the operator I and some its relatives are applied to various problems of tomography of anisotropic media, see [3, Chapters 6,7] and [2, 5].

The operator I has a big null-space in the case of $m > 0$. A symmetric tensor field can be uniquely decomposed into its solenoidal and potential parts [3, Theorem 2.6.2], and the potential part lies in the null-space. Given If , one can recover only the solenoidal part of f , and there is a reconstruction formula [3, Theorem 2.12.2]. This naturally leads to the question of what additional information should be added to the data If for the unique recovery of the entire tensor field f . One possibility is to consider the *momentum ray transforms*

$$I^k : \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n) \rightarrow \mathcal{S}(T\mathbb{S}^{n-1})$$

that are defined for $k = 0, 1, \dots$ as follows:

$$(I^k f)(x, \xi) = \int_{-\infty}^{\infty} t^k \langle f(x + t\xi), \xi^m \rangle dt \quad ((x, \xi) \in T\mathbb{S}^{n-1}). \quad (1.3)$$

In particular, $I^0 = I$. A rank m symmetric tensor field f is uniquely determined by the functions $(I^0 f, I^1 f, \dots, I^m f)$, see [3, Theorem 2.17.2] and [1].

The momentum ray transforms are primary objects of study of this paper, and we have three goals:

(1) To obtain an algorithm for recovering a rank m symmetric tensor field f from the data $(I^0 f, I^1 f, \dots, I^m f)$.

(2) To derive a version of the Reshetnyak formula [4] that expresses the norm $\|f\|_{H_t^s}$ through some norms of the functions $(I^0 f, I^1 f, \dots, I^m f)$. The H_t^s -norm is a modification of the Sobolev norm weighted differently at high and low frequencies, see Section 2 for the precise definition.

(3) To obtain stability estimates in terms of H_t^s -norms.

The first goal is achieved for arbitrary m in any dimension $n \geq 2$. The Reshetnyak formula and stability estimate are obtained in the cases of $m = 1$ and $m = 2$ only. We believe our approach works for any m , but the bulkiness of the Reshetnyak formula grows very fast with m .

The paper is organized as follows. In Section 2, we discuss basic properties of momentum ray transforms and state a few preliminaries. Section 3 presents the inversion algorithm. Section 4 is devoted to the Reshetnyak formula and stability estimates. Finally, in Section 5, we restrict ourselves to the 2-dimensional case and propose an alternate approach based on the fact that there are natural coordinates on $T\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{R}$.

2. PRELIMINARIES

2.1. Basic properties of momentum ray transforms. First of all we observe that the right-hand side of (1.3) makes sense for all $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. We define the

continuous linear operators

$$J^k : \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n) \rightarrow C^\infty(T\mathbb{S}^{n-1}) \quad (k = 0, 1, 2, \dots) \quad (2.1)$$

by

$$(J^k f)(x, \xi) = \int_{-\infty}^{\infty} t^k \langle f(x + t\xi), \xi^m \rangle dt \quad \text{for } (x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}). \quad (2.2)$$

The data $(I^0 f, I^1 f, \dots, I^m f)$ and $(J^0 f, J^1 f, \dots, J^m f)$ are equivalent as we will demonstrate right now. Therefore the operators (2.1) are also called *momentum ray transforms*. The function $J^k f$ is sometimes more convenient than $I^k f$ because the partial derivatives $\frac{\partial(J^k f)}{\partial x^i}$ and $\frac{\partial(J^k f)}{\partial \xi^i}$ are well defined. On the other hand, the function $(I^k f)(x, \xi)$ obeys good decay conditions in the second argument.

For a tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$, the function $(J^k f)(x, \xi)$ possesses the following homogeneity in the second argument

$$(J^k f)(x, t\xi) = \frac{t^{m-k}}{|t|} (J^k f)(x, \xi) \quad (0 \neq t \in \mathbb{R}) \quad (2.3)$$

and has the following property in the first argument

$$(J^k f)(x + t\xi, \xi) = \sum_{\ell=0}^k \binom{k}{\ell} (-t)^{k-\ell} (J^\ell f)(x, \xi) \quad (t \in \mathbb{R}). \quad (2.4)$$

Compare with [3, Formulas (2.1.11)–(2.1.12)]. These properties easily follow from the definition (2.2).

Comparing (1.3) and (2.2), we see that

$$I^k f = J^k f|_{T\mathbb{S}^{n-1}}. \quad (2.5)$$

On the other hand,

$$(J^k f)(x, \xi) = |\xi|^{m-2k-1} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} |\xi|^\ell \langle \xi, x \rangle^{k-\ell} (I^\ell f) \left(x - \frac{\langle \xi, x \rangle}{|\xi|^2} \xi, \frac{\xi}{|\xi|} \right) \quad (2.6)$$

as easily follows from (2.3)–(2.4). Compare with [3, Formula (2.1.13)].

Formulas (2.4) and (2.6) mean in particular that the operator I^k must always be considered together with lower order momenta (I^0, \dots, I^{k-1}) , i.e., the data $(I^0 f, \dots, I^k f)$ must always be used instead of $I^k f$.

There are two important first order differential operators on symmetric tensor fields: the *inner derivative* operator $d : C^\infty(\mathbb{R}^n; S^m \mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n; S^{m+1} \mathbb{R}^n)$ and the *divergence* operator $\delta : C^\infty(\mathbb{R}^n; S^m \mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n; S^{m-1} \mathbb{R}^n)$, see the definitions in [3, 4]. Operators I^k are related to the inner derivative by the formula

$$I^k(df) = -kI^{k-1}f$$

which is valid at least for $f \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$. This easily follows from (2.2) with the help of integration by parts. In particular, $I^k(d^\ell f) = 0$ for $k < \ell$.

Let us also mention the transformation law for operators I^k under a change of the origin in \mathbb{R}^n . Given $a \in \mathbb{R}^n$, set $f_a(x) = f(x + a)$. As easily follows from (2.5) and (2.6),

$$(I^k f_a)(x, \xi) = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \langle a, \xi \rangle^{k-\ell} (I^\ell f)(x + a - \langle a, \xi \rangle \xi, \xi) \quad \text{for } (x, \xi) \in T\mathbb{S}^{n-1}. \quad (2.7)$$

We are going to derive a formula that expresses $\|f\|_{H_t^s}$ through $(I^0 f, I^1 f, \dots, I^m f)$. Since $\|f\|_{H_t^s} = \|f_a\|_{H_t^s}$, the expression must be invariant under the transformation (2.7).

2.2. Momentum ray transforms and the Fourier transform. We use the Fourier transform $F : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$, $f \mapsto \widehat{f}$ in the following form (hereafter i is the imaginary unit and y is the Fourier dual variable of x):

$$\widehat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle y, x \rangle} f(x) dx.$$

The Fourier transform $F : \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$, $f \mapsto \widehat{f}$ on symmetric tensor fields is defined componentwise, i.e., $\widehat{f}_{i_1 \dots i_m} = \widehat{f_{i_1 \dots i_m}}$ (we use Cartesian coordinates only). Introduce also the Fourier transform $F : \mathcal{S}(T\mathbb{S}^{n-1}) \rightarrow \mathcal{S}(T\mathbb{S}^{n-1})$, $\varphi \mapsto \widehat{\varphi}$ on $T\mathbb{S}^{n-1}$ by

$$\widehat{\varphi}(y, \xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \varphi(x, \xi) dx, \quad (2.8)$$

where dx is the $(n-1)$ -dimensional Lebesgue measure on the hyperplane $\xi^\perp = \{x \in \mathbb{R}^n \mid \langle \xi, x \rangle = 0\}$. It is the standard Fourier transform in the $(n-1)$ -dimensional variable x while $\xi \in \mathbb{S}^{n-1}$ is considered as a parameter.

The Fourier transform of the momentum ray transform is given by the formula

$$\widehat{I^k f}(y, \xi) = (2\pi)^{1/2} F_{x \rightarrow y}[\langle \xi, x \rangle^k \langle f(x), \xi^m \rangle] = (2\pi)^{1/2} i^k \langle d^k \widehat{f}(y), \xi^{m+k} \rangle \quad (2.9)$$

for $(y, \xi) \in T\mathbb{S}^{n-1}$. Indeed,

$$\begin{aligned} \widehat{I^k f}(y, \xi) &= (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} (I^k f)(x, \xi) dx \\ &= (2\pi)^{(1-n)/2} \int_{\xi^\perp} \int_{-\infty}^{\infty} e^{-i\langle y, x \rangle} t^k \langle f(x + t\xi), \xi^m \rangle dt dx. \end{aligned}$$

On assuming $(y, \xi) \in T\mathbb{S}^{n-1}$, change the integration variables as $z = x + t\xi$

$$\widehat{I^k f}(y, \xi) = (2\pi)^{1/2} \langle (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle y, z \rangle} \langle \xi, z \rangle^k f(z) dz, \xi^m \rangle.$$

On using the equality

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle y, z \rangle} \langle \xi, z \rangle^k f(z) dz = i^k \langle \xi, \partial_y \rangle^k \widehat{f}(y),$$

we obtain

$$\widehat{I^k f}(y, \xi) = (2\pi)^{1/2} i^k \langle \xi, \partial_y \rangle^k \langle \widehat{f}(y), \xi^m \rangle.$$

This coincides with (2.9).

2.3. The Reshetnyak formula for scalar functions. Recall [4] that the Hilbert space $H_t^s(\mathbb{R}^n)$ is defined for $s \in \mathbb{R}$ and $t > -n/2$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H_t^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |y|^{2t} (1 + |y|^2)^{s-t} |\widehat{f}(y)|^2 dy. \quad (2.10)$$

Similarly, the Hilbert space $H_t^s(T\mathbb{S}^{n-1})$ is defined for $s \in \mathbb{R}$ and $t > -(n-1)/2$ as the completion of $\mathcal{S}(T\mathbb{S}^{n-1})$ with respect to the norm

$$\|\varphi\|_{H_t^s(T\mathbb{S}^{n-1})}^2 = \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t} (1 + |y|^2)^{s-t} |\widehat{\varphi}(y, \xi)|^2 dy d\xi, \quad (2.11)$$

where $d\xi$ is the volume form on the sphere \mathbb{S}^{n-1} induced by the Euclidean metric of \mathbb{R}^n .

The following statement is a particular case of $m = 0$ in [4, Theorem 4.2]. Given a function $f \in \mathcal{S}(\mathbb{R}^n)$, the *Reshetnyak formula*

$$\|f\|_{H_t^s(\mathbb{R}^n)}^2 = a_n \|If\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 \quad (2.12)$$

holds for every $s \in \mathbb{R}$ and every $t > -n/2$, where

$$a_n = \frac{\Gamma((n-1)/2)}{2\pi^{(n-1)/2}}. \quad (2.13)$$

(Unfortunately, there is a misprint in the formula for the coefficient $a_k = a_k(m, n)$ in Theorem 4.2 of [4]. The right formula is presented in [3, Formula (2.15.3)], the coefficients are independent of (s, t) .) Formula (2.12) will be used throughout the paper.

3. INVERSION ALGORITHM FOR MOMENTUM RAY TRANSFORMS

Let us recall the definition of the *partial symmetrization*

$$\sigma(i_1 \dots i_r) u_{i_1 \dots i_r j_1 \dots j_s} = \frac{1}{r!} \sum_{\pi \in \Pi_r} u_{i_{\pi(1)} \dots i_{\pi(r)} j_1 \dots j_s},$$

where the summation is performed over the set Π_r of all permutations of the set $\{1, \dots, r\}$.

The following statement is the main ingredient of our algorithm for recovering a rank m symmetric tensor field f from the data $(I^0 f, \dots, I^m f)$.

Theorem 3.1. *Given a tensor field $f = (f_{i_1 \dots i_m}) \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$, equalities*

$$Jf_{i_1 \dots i_m} = \frac{1}{m!} \sigma(i_1 \dots i_m) \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\partial^m (J^k f)}{\partial x^{i_1} \dots \partial x^{i_k} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_m}} \quad (3.1)$$

hold for all indices (i_1, \dots, i_m) , where the left-hand side is the result of applying the ray transform $J = J^0$ to the coordinate $f_{i_1 \dots i_m}$ considered as a scalar function on \mathbb{R}^n (we use Cartesian coordinates only).

Proof. Equality (3.1) trivially holds for $m = 0$. We proceed by induction on m . Assume (3.1) to be valid for tensor fields $f \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$ with some m .

Now assume $f \in \mathcal{S}(\mathbb{R}^n; S^{m+1} \mathbb{R}^n)$. Definition (2.2) can be written as

$$(J^k f)(x, \xi) = \xi^{p_1} \dots \xi^{p_{m+1}} \int_{-\infty}^{\infty} t^k f_{p_1 \dots p_{m+1}}(x + t\xi) dt.$$

Differentiate this equality with respect to $\xi^{i_{m+1}}$

$$\begin{aligned} \frac{\partial(J^k f)}{\partial \xi^{i_{m+1}}} &= (m+1) \xi^{p_1} \dots \xi^{p_m} \int_{-\infty}^{\infty} t^k f_{p_1 \dots p_m i_{m+1}}(x+t\xi) dt \\ &+ \frac{\partial}{\partial x^{i_{m+1}}} \left(\xi^{p_1} \dots \xi^{p_{m+1}} \int_{-\infty}^{\infty} t^{k+1} f_{p_1 \dots p_{m+1}}(x+t\xi) dt \right). \end{aligned} \quad (3.2)$$

Let us fix a value of the index i_{m+1} and define the tensor field $\tilde{f} \in \mathcal{S}(\mathbb{R}^n; S^m \mathbb{R}^n)$ by

$$\tilde{f}_{i_1 \dots i_m} = f_{i_1 \dots i_m i_{m+1}}.$$

Then (3.2) can be written as

$$\frac{\partial(J^k f)}{\partial \xi^{i_{m+1}}} = (m+1) J^k \tilde{f} + \frac{\partial(J^{k+1} f)}{\partial x^{i_{m+1}}}.$$

From this,

$$J^k \tilde{f} = \frac{1}{m+1} \left(\frac{\partial(J^k f)}{\partial \xi^{i_{m+1}}} - \frac{\partial(J^{k+1} f)}{\partial x^{i_{m+1}}} \right).$$

Differentiate this equality to obtain

$$\begin{aligned} \frac{\partial^m(J^k \tilde{f})}{\partial x^{i_1} \dots \partial x^{i_k} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_m}} &= \\ &= \frac{1}{m+1} \left(\frac{\partial^{m+1}(J^k f)}{\partial x^{i_1} \dots \partial x^{i_k} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_{m+1}}} - \frac{\partial^{m+1}(J^{k+1} f)}{\partial x^{i_1} \dots \partial x^{i_k} \partial x^{i_{m+1}} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_m}} \right). \end{aligned} \quad (3.3)$$

By the induction hypothesis,

$$J f_{i_1 \dots i_{m+1}} = J \tilde{f}_{i_1 \dots i_m} = \frac{1}{m!} \sigma(i_1 \dots i_m) \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\partial^m(J^k \tilde{f})}{\partial x^{i_1} \dots \partial x^{i_k} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_m}}.$$

Substitute value (3.3) into the last formula

$$\begin{aligned} J f_{i_1 \dots i_{m+1}} &= \frac{1}{(m+1)!} \sigma(i_1 \dots i_{m+1}) \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\partial^{m+1}(J^k f)}{\partial x^{i_1} \dots \partial x^{i_k} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_{m+1}}} \\ &- \frac{1}{(m+1)!} \sigma(i_1 \dots i_{m+1}) \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\partial^{m+1}(J^{k+1} f)}{\partial x^{i_1} \dots \partial x^{i_k} \partial x^{i_{m+1}} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_m}}. \end{aligned}$$

We have replaced the symmetrization $\sigma(i_1 \dots i_m)$ by the stronger operator $\sigma(i_1 \dots i_{m+1})$ because the left-hand side is symmetric in the indices (i_1, \dots, i_{m+1}) . In the second sum on the right-hand side, we change the summation index as $k = k' - 1$. After the change, we again use the notation k instead of k' . In such the way, we transform the formula to the form

$$\begin{aligned} J f_{i_1 \dots i_{m+1}} &= \frac{1}{(m+1)!} \sigma(i_1 \dots i_{m+1}) \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\partial^{m+1}(J^k f)}{\partial x^{i_1} \dots \partial x^{i_k} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_{m+1}}} \\ &+ \frac{1}{(m+1)!} \sigma(i_1 \dots i_{m+1}) \sum_{k=1}^{m+1} (-1)^k \binom{m}{k-1} \frac{\partial^{m+1}(J^k f)}{\partial x^{i_1} \dots \partial x^{i_{k-1}} \partial x^{i_{m+1}} \partial \xi^{i_k} \dots \partial \xi^{i_m}}. \end{aligned}$$

We assume binomial coefficients $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ to be defined for all integers m and k under the agreement: $\binom{m}{k} = 0$ if either $m < 0$ or $k < 0$ or $k > m$. Therefore both

summations can be extended to the limits $0 \leq k \leq m+1$. Besides this, we can write indices (i_1, \dots, i_{m+1}) in an arbitrary order on the right-hand side because of the presence of the symmetrization $\sigma(i_1 \dots i_{m+1})$. With the help of the Pascal relation $\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}$, the formula takes the form

$$Jf_{i_1 \dots i_{m+1}} = \frac{1}{(m+1)!} \sigma(i_1 \dots i_{m+1}) \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \frac{\partial^{m+1}(J^k f)}{\partial x^{i_1} \dots \partial x^{i_k} \partial \xi^{i_{k+1}} \dots \partial \xi^{i_{m+1}}}.$$

This finishes the induction step. \square

Let us recall the inversion formula for recovering a scalar function $f \in \mathcal{S}(\mathbb{R}^n)$ from If :

$$f(x) = \frac{\Gamma((n-1)/2)}{4\pi^{(n+1)/2}} (-\Delta)^{1/2} \int_{\mathbb{S}^{n-1}} (If)(x - \langle \xi, x \rangle \xi, \xi) d\xi. \quad (3.4)$$

This is a particular case of $m = 0$ of [3, Theorem 2.12.2].

Our algorithm for recovering a symmetric m -tensor field f from the collection of functions $(I^0 f, \dots, I^m f)$ is as follows. Given $(I^0 f, \dots, I^m f)$, we find the data $(J^0 f, \dots, J^m f)$ by formula (2.6). Then we find the functions $Jf_{i_1 \dots i_m}$ for all values of indices by (3.1) and find $If_{i_1 \dots i_m} = Jf_{i_1 \dots i_m}|_{T\mathbb{S}^{n-1}}$. From the latter data we recover all components $f_{i_1 \dots i_m}$ of the field f by formula (3.4).

The stability of the recovery procedure for a scalar function is completely described by the Reshetnyak formula (2.12). For higher rank tensor fields, the stability question is more delicate because of the presence of m^{th} order derivatives in formula (3.1). We will investigate the stability question in the next section.

4. RESHETNYAK FORMULA FOR MOMENTUM RAY TRANSFORMS

In this section, we derive the Reshetnyak formula and stability estimate for $m = 1$ and for $m = 2$.

4.1. Operators X_i and Ξ_i . In the cases of $m = 1$ and of $m = 2$, formula (3.1) takes the forms

$$Jf_i = \frac{\partial(J^0 f)}{\partial \xi^i} - \frac{\partial(J^1 f)}{\partial x^i} \quad (4.1)$$

and

$$Jf_{ij} = \frac{1}{2} \left(\frac{\partial^2(J^0 f)}{\partial \xi^i \partial \xi^j} - \frac{\partial^2(J^1 f)}{\partial x^i \partial \xi^j} - \frac{\partial^2(J^1 f)}{\partial x^j \partial \xi^i} + \frac{\partial^2(J^2 f)}{\partial x^i \partial x^j} \right) \quad (4.2)$$

respectively. We are going to rewrite these formulas in intrinsic terms of the manifold $T\mathbb{S}^{n-1}$.

Introduce the vector fields on $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi \neq 0\}$

$$\begin{aligned} \tilde{X}_i &= \frac{\partial}{\partial x^i} - \xi_i \xi^p \frac{\partial}{\partial x^p} \quad (1 \leq i \leq n), \\ \tilde{\Xi}_i &= \frac{\partial}{\partial \xi^i} - x_i \xi^p \frac{\partial}{\partial x^p} - \xi_i \xi^p \frac{\partial}{\partial \xi^p} \quad (1 \leq i \leq n). \end{aligned} \quad (4.3)$$

The notations \tilde{X}_i and $\tilde{\Xi}_i$ are chosen because the derivatives $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial \xi^i}$ are in some sense leading terms on the right-hand sides of (4.3).

Lemma 4.1. *At every point $(x, \xi) \in T\mathbb{S}^{n-1}$, vectors $\tilde{X}_i(x, \xi)$ and $\tilde{\Xi}_i(x, \xi)$ ($1 \leq i \leq n$) are tangent to $T\mathbb{S}^{n-1}$. Let X_i and Ξ_i be the restrictions of vector fields \tilde{X}_i and $\tilde{\Xi}_i$ to the manifold $T\mathbb{S}^{n-1}$ respectively. Thus, X_i and Ξ_i are smooth vector fields on $T\mathbb{S}^{n-1}$ and can be considered as first order differential operators*

$$X_i, \Xi_i : C^\infty(T\mathbb{S}^{n-1}) \rightarrow C^\infty(T\mathbb{S}^{n-1}).$$

The operators satisfy

$$[X_i, X_j] = 0, \quad (4.4)$$

$$[\Xi_i, \Xi_j] = x_i X_j - x_j X_i + \xi_i \Xi_j - \xi_j \Xi_i, \quad (4.5)$$

$$[X_i, \Xi_j] = \xi_i X_j. \quad (4.6)$$

At every point $(x, \xi) \in T\mathbb{S}^{n-1}$, vectors $X_i(x, \xi), \Xi_i(x, \xi)$ ($1 \leq i \leq n$) generate the tangent space $T_{(x, \xi)}(T\mathbb{S}^{n-1})$ and satisfy

$$\xi^i X_i(x, \xi) = 0, \quad \xi^i \Xi_i(x, \xi) = 0. \quad (4.7)$$

Proof. Definition (4.3) implies

$$\tilde{X}_i |\xi|^2 = 0, \quad \tilde{\Xi}_i |\xi|^2 = 2\xi_i (1 - |\xi|^2), \quad \tilde{X}_i \langle \xi, x \rangle = \xi_i (1 - |\xi|^2), \quad \tilde{\Xi}_i \langle \xi, x \rangle = x_i (1 - |\xi|^2) - \xi_i \langle \xi, x \rangle.$$

Right-hand sides of these equalities vanish on $T\mathbb{S}^{n-1}$. This proves the first statement.

From definition (4.3),

$$\xi^i \tilde{X}_i = (1 - |\xi|^2) \xi^p \frac{\partial}{\partial x^p}, \quad \xi^i \tilde{\Xi}_i = \langle \xi, x \rangle \xi^p \frac{\partial}{\partial x^p} - (1 - |\xi|^2) \xi^p \frac{\partial}{\partial \xi^p}.$$

Right-hand sides of these equalities vanish on $T\mathbb{S}^{n-1}$. This proves (4.7).

Recall the well known formula:

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

for vector fields X, Y and for functions f, g . On using this formula, one easily derives from definition (4.3)

$$[\tilde{X}_i, \tilde{\Xi}_j] = \xi_i \frac{\partial}{\partial x^j} - \xi_i \xi_j \xi^p \frac{\partial}{\partial x^p}. \quad (4.8)$$

On the other hand,

$$\xi_i \tilde{X}_j = \xi_i \left(\frac{\partial}{\partial x^j} - \xi_i \xi^p \frac{\partial}{\partial x^p} \right) = \xi_i \frac{\partial}{\partial x^j} - \xi_i \xi_j \xi^p \frac{\partial}{\partial x^p}. \quad (4.9)$$

Comparing (4.8) and (4.9), we see that $[\tilde{X}_i, \tilde{\Xi}_j] = \xi_i \tilde{X}_j$. This proves (4.6). Formulas (4.4)–(4.5) are proved in a similar way.

Finally we prove that, at a point $(x, \xi) \in T\mathbb{S}^{n-1}$, the vectors $X_i(x, \xi), \Xi_i(x, \xi)$ ($1 \leq i \leq n$) generate the tangent space $T_{(x, \xi)}(T\mathbb{S}^{n-1})$. To this end we have to demonstrate that any linear dependence between these vectors is actually a corollary of (4.7). Assume that

$$\alpha^i X_i + \beta^i \Xi_i = 0 \quad (4.10)$$

with some coefficients α^i, β^i . Substitute values (4.3) into this equality

$$\alpha^i \left(\frac{\partial}{\partial x^i} - \xi_i \xi^p \frac{\partial}{\partial x^p} \right) + \beta^i \left(\frac{\partial}{\partial \xi^i} - x_i \xi^p \frac{\partial}{\partial x^p} - \xi_i \xi^p \frac{\partial}{\partial \xi^p} \right) = 0.$$

This can be written in the form

$$(\alpha^i (\delta_i^p - \xi_i \xi^p) - \beta^i x_i \xi^p) \frac{\partial}{\partial x^p} + \beta^i (\delta_i^p - \xi_i \xi^p) \frac{\partial}{\partial \xi^p} = 0,$$

where δ_j^i is the Kronecker tensor. Since vectors $\frac{\partial}{\partial x^p}, \frac{\partial}{\partial \xi^p}$ ($p = 1, \dots, n$) are linearly independent, the last equation implies

$$(\delta_i^p - \xi_i \xi^p) \alpha^i - x_i \xi^p \beta^i = 0 \quad (p = 1, \dots, n), \quad (4.11)$$

$$(\delta_i^p - \xi_i \xi^p) \beta^i = 0 \quad (p = 1, \dots, n). \quad (4.12)$$

At a point $(x, \xi) \in T\mathbb{S}^{n-1}$, the rank of the matrix $(\delta_i^p - \xi_i \xi^p)$ of system (4.12) is equal to $n - 1$ and any solution to the system is of the form $\beta^i = \beta_0 \xi^i$. System (4.11) takes now the form

$$(\delta_i^p - \xi_i \xi^p) \alpha^i = 0 \quad (p = 1, \dots, n).$$

As before, this implies $\alpha^i = \alpha_0 \xi^i$. Thus, equation (4.10) is actually of the form

$$\alpha_0 \xi^i X_i + \beta_0 \xi^i \Xi_i = 0.$$

This means that any linear dependence between vectors $X_i(x, \xi), \Xi_i(x, \xi)$ ($1 \leq i \leq n$) is actually a corollary of (4.7). \square

We can now present the intrinsic forms of equations (4.1) and (4.2).

Theorem 4.2. *For a vector field $f = (f_i) \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^n)$, the equality*

$$If_i = (\Xi_i + \xi_i)(I^0 f) - X_i(I^1 f) \quad (4.13)$$

holds on $T\mathbb{S}^{n-1}$ for every i , where the left-hand side is the result of applying the ray transform $I = I^0$ to the coordinates f_i considered as scalar functions on \mathbb{R}^n .

Theorem 4.3. *For a tensor field $f = (f_{ij}) \in \mathcal{S}(\mathbb{R}^n; S^2\mathbb{R}^n)$, the equality*

$$If_{ij} = \frac{1}{2} \sigma(ij) \left[X_i X_j (I^2 f) - 2 \Xi_i X_j (I^1 f) - 4 \xi_i X_j (I^1 f) + \Xi_i \Xi_j (I^0 f) + x_i X_j (I^0 f) + 3 \xi_i \Xi_j (I^0 f) + 3 \delta_{ij} (I^0 f) - \xi_i \xi_j (I^0 f) \right] \quad (4.14)$$

holds on $T\mathbb{S}^{n-1}$ for all indices (i, j) , where δ_{ij} is the Kronecker tensor and the left-hand side is the result of applying the ray transform $I = I^0$ to the coordinates f_{ij} considered as scalar functions on \mathbb{R}^n .

We present the proof of Theorem 4.2. Theorem 4.3 is proved in the same way although all involved calculations are more cumbersome.

Proof of Theorem 4.2. By the very definition of X_i and Ξ_i , the equalities

$$X_i \varphi = \tilde{X}_i \psi, \quad \Xi_i \varphi = \tilde{\Xi}_i \psi$$

hold on $T\mathbb{S}^{n-1}$ for an arbitrary function $\varphi \in C^\infty(T\mathbb{S}^{n-1})$, where ψ is an arbitrary smooth extension of φ to some neighborhood of $T\mathbb{S}^{n-1}$ in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. For every k , the function $J^k f$ is an extension of $I^k f$. Therefore

$$X_i(I^k f) = \tilde{X}_i(J^k f), \quad \Xi_i(I^k f) = \tilde{\Xi}_i(J^k f) \quad \text{on } T\mathbb{S}^{n-1}.$$

Substitute values (4.3) to obtain

$$\left. \begin{aligned} X_i(I^k f) &= \left(\frac{\partial}{\partial x^i} - \xi_i \xi^p \frac{\partial}{\partial x^p} \right) (J^k f), \\ \Xi_i(I^k f) &= \left(\frac{\partial}{\partial \xi^i} - x_i \xi^p \frac{\partial}{\partial x^p} - \xi_i \xi^p \frac{\partial}{\partial \xi^p} \right) (J^k f) \end{aligned} \right\} \quad \text{on } T\mathbb{S}^{n-1}; \quad (4.15)$$

Let us specify (4.15) for $k = 0$. As is seen from (2.3), the function $(J^0f)(x, \xi)$ is positively homogeneous of zero degree in the second argument. By the Euler equation for homogeneous functions,

$$\xi^p \frac{\partial(J^0f)}{\partial \xi^p} = 0. \quad (4.16)$$

Besides this, J^0f satisfies (see (2.4))

$$(J^0f)(x + t\xi, \xi) = (J^0f)(x, \xi) \quad (t \in \mathbb{R}).$$

Differentiating this identity with respect to t and then setting $t = 0$, we obtain

$$\xi^p \frac{\partial(J^0f)}{\partial x^p} = 0. \quad (4.17)$$

In virtue of (4.16)–(4.17), equalities (4.15) for $k = 0$ are simplified to the following ones:

$$\frac{\partial(J^0f)}{\partial x^i} = X_i(I^0f), \quad \frac{\partial(J^0f)}{\partial \xi^i} = \Xi_i(I^0f) \quad \text{on } T\mathbb{S}^{n-1}. \quad (4.18)$$

Let us specify (4.15) for $k = 1$. As is seen from (2.3), the function $(J^1f)(x, \xi)$ is positively homogeneous of degree -1 in the second argument. By the Euler equation for homogeneous functions,

$$\xi^p \frac{\partial(J^1f)}{\partial \xi^p} = -J^1f. \quad (4.19)$$

Besides this, J^1f satisfies (see (2.4))

$$(J^1f)(x + t\xi, \xi) = -t(J^0f)(x + t\xi, \xi) + (J^1f)(x, \xi) \quad (t \in \mathbb{R}).$$

Differentiating this identity with respect to t and then setting $t = 0$, we obtain

$$\xi^p \frac{\partial(J^1f)}{\partial x^p} = -(J^0f). \quad (4.20)$$

In virtue of (4.19)–(4.20), equalities (4.15) for $k = 1$ are simplified to the following ones:

$$\frac{\partial(J^1f)}{\partial x^i} = X_i(I^1f) - \xi_i(I^0f), \quad \frac{\partial(J^1f)}{\partial \xi^i} = (\Xi_i - \xi_i)(I^1f) - x_i(I^0f) \quad \text{on } T\mathbb{S}^{n-1}. \quad (4.21)$$

Inserting value (4.18) of $\frac{\partial(J^0f)}{\partial \xi^i}$ and value (4.21) of $\frac{\partial(J^1f)}{\partial x^i}$ into (4.1), we arrive to (4.13). \square

Recall that y_i is the Fourier dual variable of x_i . The formulas for commuting the Fourier transform with operators X_i and Ξ_i are described by the following

Lemma 4.4. *The equalities*

$$\widehat{X_i \varphi} = i y_i \widehat{\varphi}, \quad \widehat{\Xi_i \varphi} = \Xi_i \widehat{\varphi}, \quad \widehat{x_i \varphi} = i X_i \widehat{\varphi} \quad (4.22)$$

hold for every function $\varphi \in \mathcal{S}(T\mathbb{S}^{n-1})$ and for every i , $1 \leq i \leq n$.

Proof. Given a function $\varphi \in \mathcal{S}(T\mathbb{S}^{n-1})$, define $\psi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ by

$$\psi(x, \xi) = \varphi\left(x - \frac{\langle \xi, x \rangle}{|\xi|^2} \xi, \frac{\xi}{|\xi|}\right).$$

The function satisfies the identities

$$\psi(x, t\xi) = \psi(x, \xi) \quad (0 \neq t \in \mathbb{R}), \quad \psi(x, x + t\xi) = \psi(x, \xi) \quad (t \in \mathbb{R})$$

which imply (by the same arguments that were used for deriving (4.16)–(4.17))

$$\xi^p \frac{\partial \psi}{\partial x^p} = 0, \quad \xi^p \frac{\partial \psi}{\partial \xi^p} = 0. \quad (4.23)$$

For $(y, \xi) \in T\mathbb{S}^{n-1}$, the definition (2.8) of the Fourier transform can be written in terms of ψ :

$$\widehat{\varphi}(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \psi(x, \xi) dx. \quad (4.24)$$

This formula makes sense for any $(y, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ sufficiently close to the submanifold $T\mathbb{S}^{n-1} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Therefore the partial derivatives $\frac{\partial \widehat{\varphi}}{\partial y^i}$ and $\frac{\partial \widehat{\varphi}}{\partial \xi^i}$ make sense.

By (4.3),

$$X_i \varphi = \widetilde{X}_i \psi = \frac{\partial \psi}{\partial x^i} - \xi_i \xi^p \frac{\partial \psi}{\partial x^p}.$$

With the help of (4.23), this is simplified to

$$X_i \varphi = \frac{\partial \psi}{\partial x^i}.$$

Applying the Fourier transform to this equality, we obtain

$$\widehat{X_i \varphi}(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \frac{\partial \psi}{\partial x^i}(x, \xi) dx.$$

From this with the help of integration by parts

$$\widehat{X_i \varphi}(y, \xi) = i y_i (2\pi)^{-(n-1)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \psi(x, \xi) dx = i y_i \widehat{\varphi}(y, \xi).$$

This proves the first of formulas (4.22).

Differentiate equality (4.24) with respect to y^i

$$\frac{\partial \widehat{\varphi}}{\partial y^i}(y, \xi) = -i (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} x_i \psi(x, \xi) dx. \quad (4.25)$$

From this

$$\xi^p \frac{\partial \widehat{\varphi}}{\partial y^p}(y, \xi) = -i (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \langle \xi, x \rangle \psi(x, \xi) dx = 0. \quad (4.26)$$

By (4.3),

$$\Xi_i \widehat{\varphi} = \frac{\partial \widehat{\varphi}}{\partial \xi^i} - y_i \xi^p \frac{\partial \widehat{\varphi}}{\partial y^p} - \xi_i \xi^p \frac{\partial \widehat{\varphi}}{\partial \xi^p}.$$

In virtue of (4.26), this is simplified to

$$\Xi_i \widehat{\varphi} = \frac{\partial \widehat{\varphi}}{\partial \xi^i} - \xi_i \xi^p \frac{\partial \widehat{\varphi}}{\partial \xi^p}. \quad (4.27)$$

The differentiation of equality (4.24) with respect to ξ^i is not very easy because the integration hyperplane ξ^\perp depends on ξ . We first transform integral (4.24) into an integral over a hyperplane independent of ξ . Fix a vector $\xi_0 \in \mathbb{S}^{n-1}$. For an arbitrary vector $\xi \in \mathbb{S}^{n-1}$ sufficiently close to ξ_0 , the orthogonal projection

$$\xi_0^\perp \rightarrow \xi^\perp, \quad x' \mapsto x = x' - \langle \xi, x' \rangle \xi \quad (4.28)$$

is one-to-one. We change the integration variable in (4.24) according to (4.28). The Jacobian of the change is $\langle \xi_0, \xi \rangle^{-1}$. After the change, formula (4.24) takes the form

$$\hat{\varphi}(y, \xi) = \frac{(2\pi)^{(1-n)/2}}{\langle \xi_0, \xi \rangle} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} \psi(x' - \langle \xi, x' \rangle \xi, \xi) dx'.$$

We can now differentiate this equality with respect to ξ^i

$$\begin{aligned} \frac{\partial \hat{\varphi}}{\partial \xi^i}(y, \xi) &= -\frac{\xi_{0,i}}{\langle \xi_0, \xi \rangle^2} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} \psi(x' - \langle \xi, x' \rangle \xi, \xi) dx' \\ &\quad + \frac{1}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} \frac{\partial e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle}}{\partial \xi^i} \psi(x' - \langle \xi, x' \rangle \xi, \xi) dx' \\ &\quad + \frac{1}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} \frac{\partial \psi(x' - \langle \xi, x' \rangle \xi, \xi)}{\partial \xi^i} dx'. \end{aligned}$$

Substituting the values

$$\frac{\partial e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle}}{\partial \xi^i} = i e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} (\langle \xi, y \rangle x'_i + \langle \xi, x' \rangle y_i)$$

and

$$\begin{aligned} \frac{\partial \psi(x' - \langle \xi, x' \rangle \xi, \xi)}{\partial \xi^i} &= \frac{\partial \psi}{\partial \xi^i}(x' - \langle \xi, x' \rangle \xi, \xi) - \langle \xi, x' \rangle \frac{\partial \psi}{\partial x^i}(x' - \langle \xi, x' \rangle \xi, \xi) \\ &\quad - x'_i \xi^p \frac{\partial \psi}{\partial x^p}(x' - \langle \xi, x' \rangle \xi, \xi), \end{aligned}$$

we obtain

$$\begin{aligned} \frac{\partial \hat{\varphi}}{\partial \xi^i}(y, \xi) &= -\frac{\xi_{0,i}}{\langle \xi_0, \xi \rangle^2} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} \psi(x' - \langle \xi, x' \rangle \xi, \xi) dx' \\ &\quad + \frac{i}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} (\langle \xi, y \rangle x'_i + \langle \xi, x' \rangle y_i) \psi(x' - \langle \xi, x' \rangle \xi, \xi) dx' \\ &\quad + \frac{1}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} \frac{\partial \psi}{\partial \xi^i}(x' - \langle \xi, x' \rangle \xi, \xi) dx' \\ &\quad - \frac{1}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} \langle \xi, x' \rangle \frac{\partial \psi}{\partial x^i}(x' - \langle \xi, x' \rangle \xi, \xi) dx' \\ &\quad - \frac{1}{\langle \xi_0, \xi \rangle} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' - \langle \xi, x' \rangle \xi \rangle} x'_i \xi^p \frac{\partial \psi}{\partial x^p}(x' - \langle \xi, x' \rangle \xi, \xi) dx'. \end{aligned}$$

On assuming $(y, \xi_0) \in T\mathbb{S}^{n-1}$, we set $\xi = \xi_0$ in the latter formula. The formula simplifies to the following one:

$$\begin{aligned} \frac{\partial \hat{\varphi}}{\partial \xi^i}(y, \xi_0) &= -\xi_{0,i} (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' \rangle} \psi(x', \xi_0) dx' \\ &\quad + (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' \rangle} \frac{\partial \psi}{\partial \xi^i}(x', \xi_0) dx' \\ &\quad - (2\pi)^{(1-n)/2} \int_{\xi_0^\perp} e^{-i\langle y, x' \rangle} x'_i \xi_0^p \frac{\partial \psi}{\partial x^p}(x', \xi_0) dx'. \end{aligned}$$

By (4.23), the integrand of the last integral is identically equal to zero. Thus, replacing the notations ξ_0 and x' with ξ and x respectively, we obtain

$$\frac{\partial \hat{\varphi}}{\partial \xi^i}(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \frac{\partial \psi}{\partial \xi^i}(x, \xi) dx - \xi_i \hat{\varphi}(y, \xi) \quad \text{for } (y, \xi) \in T\mathbb{S}^{n-1}. \quad (4.29)$$

With the help of (2.8), we obtain from (4.29)

$$\xi^p \frac{\partial \hat{\varphi}}{\partial \xi^p}(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \xi^p \frac{\partial \psi}{\partial \xi^p}(x, \xi) dx - \hat{\varphi}(y, \xi) = -\hat{\varphi}(y, \xi) \quad \text{for } (y, \xi) \in T\mathbb{S}^{n-1}. \quad (4.30)$$

Substitute values (4.29) and (4.30) into (4.27) to obtain

$$(\Xi_i \hat{\varphi})(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} \frac{\partial \psi}{\partial \xi^i}(x, \xi) dx \quad \text{for } (y, \xi) \in T\mathbb{S}^{n-1}. \quad (4.31)$$

By (4.23),

$$(\tilde{\Xi}_i \psi)(x, \xi) = \frac{\partial \psi}{\partial \xi^i}(x, \xi) \quad \text{for } (x, \xi) \in T\mathbb{S}^{n-1}.$$

Therefore formula (4.31) can be written as

$$(\Xi_i \hat{\varphi})(y, \xi) = (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} (\tilde{\Xi}_i \psi)(x, \xi) dx = \widehat{\Xi_i \varphi}(y, \xi) \quad \text{for } (y, \xi) \in T\mathbb{S}^{n-1}.$$

This proves the second of formulas (4.22).

By the definition of X_i ,

$$X_i \hat{\varphi} = \frac{\partial \hat{\varphi}}{\partial y^i} - \xi_i \xi^p \frac{\partial \hat{\varphi}}{\partial y^p}.$$

In virtue of (4.26), this is simplified to

$$X_i \hat{\varphi} = \frac{\partial \hat{\varphi}}{\partial y^i}.$$

With the help of (4.25), this gives

$$X_i \hat{\varphi} = -i (2\pi)^{(1-n)/2} \int_{\xi^\perp} e^{-i\langle y, x \rangle} x_i \varphi(x, \xi) dx = -i \widehat{x_i \varphi}.$$

This proves the last of formulas (4.22). □

4.2. Reshetnyak formula for vector fields.

Theorem 4.5. *Given a vector field $f \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^n)$, the equality*

$$\begin{aligned} \|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)}^2 &= a_n \left[\sum_{i=1}^n \|\Xi_i(I^0 f)\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 + \|I^0 f\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 \right. \\ &\quad \left. + \|I^1 f\|_{H_{t+3/2}^{s+3/2}(T\mathbb{S}^{n-1})}^2 + 2\Re(iZ(I^0 f), I^1 f)_{H_{t+1}^{s+1}(T\mathbb{S}^{n-1})} \right] \end{aligned} \quad (4.32)$$

holds for every $s \in \mathbb{R}$ and every $t > -n/2$, where the constant a_n is defined by (2.13) and Z is the first order pseudodifferential operator

$$Z : \mathcal{S}(T\mathbb{S}^{n-1}) \rightarrow C^\infty(T\mathbb{S}^{n-1})$$

defined by

$$\widehat{Z}\varphi(y, \xi) = \frac{y^i}{|y|} (\Xi_i \widehat{\varphi})(y, \xi). \quad (4.33)$$

Proof. Apply the Fourier transform to formula (4.13)

$$\widehat{I}f_i = -\widehat{X}_i(\widehat{I}f) + \widehat{\Xi}_i(\widehat{I}^0 f) + \xi_i \widehat{I}^0 f.$$

With the help of Lemma 4.4, this gives

$$\widehat{I}f_i = -iy_i \widehat{I}^1 f + \widehat{\Xi}_i \widehat{I}^0 f + \xi_i \widehat{I}^0 f. \quad (4.34)$$

For a vector field $f = (f_i)$,

$$\|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)}^2 = \sum_{i=1}^n \|f_i\|_{H_t^s(\mathbb{R}^n)}^2.$$

Applying the Reshetnyak formula (2.12) for scalar functions, we obtain

$$\|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)}^2 = a_n \sum_{i=1}^n \|\widehat{I}f_i\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2.$$

On using the definition (2.11) of the $H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})$ -norm, this is written as

$$\|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)}^2 = \frac{a_n}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} \sum_{i=1}^n |\widehat{I}f_i(y, \xi)|^2 dy d\xi. \quad (4.35)$$

Now, we calculate the integrand on (4.35). By (4.34),

$$\sum_{i=1}^n |\widehat{I}f_i|^2 = \sum_{i=1}^n \left(\widehat{\Xi}_i \widehat{I}^0 f + \xi_i \widehat{I}^0 f - iy_i \widehat{I}^1 f \right) \left(\widehat{\Xi}_i \overline{\widehat{I}^0 f} + \xi_i \overline{\widehat{I}^0 f} + iy_i \overline{\widehat{I}^1 f} \right).$$

After opening parentheses and using the equalities $|\xi|^2 = 1$, $\langle \xi, y \rangle = 0$, this gives

$$\sum_{i=1}^n |\widehat{I}f_i|^2 = \sum_{i=1}^n |\widehat{\Xi}_i \widehat{I}^0 f|^2 + |\widehat{I}^0 f|^2 + |y|^2 |\widehat{I}^1 f|^2 + 2\Re((\xi^i \widehat{\Xi}_i) \widehat{I}^0 f \cdot \overline{\widehat{I}^0 f}) + 2\Re(i(y^i \widehat{\Xi}_i) \widehat{I}^0 f \cdot \overline{\widehat{I}^1 f}).$$

By (4.7), $\xi^i \widehat{\Xi}_i = 0$, and the formula takes the form

$$\sum_{i=1}^n |\widehat{I}f_i|^2 = \sum_{i=1}^n |\widehat{\Xi}_i \widehat{I}^0 f|^2 + |\widehat{I}^0 f|^2 + |y|^2 |\widehat{I}^1 f|^2 + 2\Re(i(y^i \widehat{\Xi}_i) \widehat{I}^0 f \cdot \overline{\widehat{I}^1 f}).$$

On using the operator (4.33), this can be written as

$$\sum_{i=1}^n |\widehat{I}f_i|^2 = \sum_{i=1}^n |\Xi_i \widehat{I^0 f}|^2 + |\widehat{I^0 f}|^2 + |y|^2 |\widehat{I^1 f}|^2 + 2|y| \Re(i \widehat{Z}(\widehat{I^0 f}) \cdot \overline{\widehat{I^1 f}}). \quad (4.36)$$

Substituting the value (4.36) for $\sum_{i=1}^n |\widehat{I}f_i|^2$ into (4.35), we get,

$$\begin{aligned} \|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)}^2 &= a_n \left[\sum_{i=1}^n \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} |\Xi_i \widehat{I^0 f}(y, \xi)|^2 dy d\xi \right. \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} |\widehat{I^0 f}(y, \xi)|^2 dy d\xi \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+3} (1 + |y|^2)^{s-t} |\widehat{I^1 f}(y, \xi)|^2 dy d\xi \\ &\quad \left. + 2\Re \left(\frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+2} (1 + |y|^2)^{s-t} i \widehat{Z}(\widehat{I^0 f})(y, \xi) \overline{\widehat{I^1 f}(y, \xi)} dy d\xi \right) \right]. \end{aligned}$$

On using the statement $\Xi_i \widehat{I^0 f} = \widehat{\Xi_i(I^0 f)}$ of Lemma 4.4, we rewrite the latter formula as

$$\begin{aligned} \|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)}^2 &= a_n \left[\sum_{i=1}^n \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} |\widehat{\Xi_i(I^0 f)}(y, \xi)|^2 dy d\xi \right. \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} |\widehat{I^0 f}(y, \xi)|^2 dy d\xi \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+3} (1 + |y|^2)^{s-t} |\widehat{I^1 f}(y, \xi)|^2 dy d\xi \\ &\quad \left. + 2\Re \left(\frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+2} (1 + |y|^2)^{s-t} i \widehat{Z}(\widehat{I^0 f})(y, \xi) \overline{\widehat{I^1 f}(y, \xi)} dy d\xi \right) \right]. \end{aligned}$$

In view of (2.11), this is equivalent to (4.32). \square

Formula (4.32) suggests the idea of introducing the norm

$$\begin{aligned} \|(\varphi, \psi)\|_{\mathcal{H}_t^{1,s}(T\mathbb{S}^{n-1})}^2 &= a_n \left[\sum_{i=1}^n \|\Xi_i \varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 + \|\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 \right. \\ &\quad \left. + \|\psi\|_{H_{t+3/2}^{s+3/2}(T\mathbb{S}^{n-1})}^2 + 2\Re(i Z \varphi, \psi)_{H_{t+1}^{s+1}(T\mathbb{S}^{n-1})} \right] \end{aligned} \quad (4.37)$$

on the space $\mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1})$. Let us demonstrate that the right-hand side of (4.37) is positive for $(\varphi, \psi) \neq (0, 0)$. Indeed, the inequality

$$\left| (\varphi, \psi)_{H_{t+1}^{s+1}(T\mathbb{S}^{n-1})} \right| \leq \|\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})} \|\psi\|_{H_{t+3/2}^{s+3/2}(T\mathbb{S}^{n-1})} \quad (4.38)$$

easily follows from the definition of H_t^s -norms with the help of the Schwartz inequality. In particular,

$$\begin{aligned} \left| (iZ\varphi, \psi)_{H_{t+1}^{s+1}(T\mathbb{S}^{n-1})} \right| &\leq \|Z\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})} \|\psi\|_{H_{t+3/2}^{s+3/2}(T\mathbb{S}^{n-1})} \\ &\leq \frac{1}{2} \|Z\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 + \frac{1}{2} \|\psi\|_{H_{t+3/2}^{s+3/2}(T\mathbb{S}^{n-1})}^2. \end{aligned}$$

The inequality

$$\|Z\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 \leq \sum_{i=1}^n \|\Xi_i\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2$$

easily follows from definition (4.33) of the operator Z . Together with the previous inequality, it gives

$$2 \left| (iZ\varphi, \psi)_{H_{t+1}^{s+1}(T\mathbb{S}^{n-1})} \right| \leq \sum_{i=1}^n \|\Xi_i\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 + \|\psi\|_{H_{t+3/2}^{s+3/2}(T\mathbb{S}^{n-1})}^2. \quad (4.39)$$

On using the last inequality, we derive from (4.37)

$$\|(\varphi, \psi)\|_{\mathcal{H}_t^{1,s}(T\mathbb{S}^{n-1})}^2 \geq a_n \|\varphi\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2.$$

The right-hand side of this inequality is non-negative and equals zero only if $\varphi = 0$. In the latter case the right-hand side of (4.37) is non-negative and equals zero only if $\psi = 0$.

We define the Hilbert space $\mathcal{H}_t^{1,s}(T\mathbb{S}^{n-1})$ as the completion of $\mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1})$ with respect to the norm (4.37).

Corollary 4.6. *The map*

$$\mathcal{S}(\mathbb{R}^n; \mathbb{C}^n) \rightarrow \mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1}), \quad f \mapsto (I^0f, I^1f)$$

extends to the linear isometric embedding

$$H_t^s(\mathbb{R}^n; \mathbb{C}^n) \rightarrow \mathcal{H}_t^{1,s}(T\mathbb{S}^{n-1}) \quad (4.40)$$

for every $s \in \mathbb{R}$ and every $t > -n/2$. In other words, the equality

$$\|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)} = \|(I^0f, I^1f)\|_{\mathcal{H}_t^{1,s}(T\mathbb{S}^{n-1})} \quad (4.41)$$

holds for all $f \in H_t^s(\mathbb{R}^n; \mathbb{C}^n)$.

The embedding (4.40) is not surjective because the function $(I^0f)(x, \xi)$ is odd in the second argument and the function $(I^1f)(x, \xi)$ is even in the second argument as is seen from (2.3). The interesting open question is about a description of the range of the operator (4.40).

Formula (4.41) gives the best stability estimate for the problem of recovering a vector field f from the data (I^0f, I^1f) , although a rather exotic norm is involved. The stability estimate involving more traditional norms can be also easily derived from the Reshetnyak formula.

Corollary 4.7. *For every $s \in \mathbb{R}$ and for every $t > -n/2$, the stability estimate*

$$\|f\|_{H_t^s(\mathbb{R}^n; \mathbb{C}^n)}^2 \leq 2a_n \left(\sum_{i=1}^n \|\Xi_i(I^0f)\|_{H_{t+1/2}^{s+1/2}}^2 + \|I^0f\|_{H_{t+1/2}^{s+1/2}}^2 + \|I^1f\|_{H_{t+3/2}^{s+3/2}}^2 \right). \quad (4.42)$$

holds for all $f \in H_t^s(\mathbb{R}^n; \mathbb{C}^n)$.

The abbreviated notation $\|\cdot\|_{H_t^s}$ for the norm $\|\cdot\|_{H_t^s(T\mathbb{S}^{n-1})}$ is used on the right-hand side of (4.42). Inequality (4.42) follows from (4.32) with the help of (4.39).

4.3. **Reshetnyak formula for second rank symmetric tensor fields.** Along with the operator Z defined by (4.33), we need two similar operators

$$Q, W : \mathcal{S}(T\mathbb{S}^{n-1}) \rightarrow C^\infty(T\mathbb{S}^{n-1})$$

that are defined by the formulas

$$(y^i y^j \Xi_i \Xi_j \widehat{\varphi})(y, \xi) = |y|^2 \widehat{Q\varphi}(y, \xi)$$

and

$$(y^i X_i \widehat{\varphi})(y, \xi) = |y| \widehat{W\varphi}(y, \xi)$$

respectively. We will sometimes write Ξ^i instead of Ξ_i in order to adopt our formulas to the Einstein summation rule.

Theorem 4.8. *Given a tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^2\mathbb{R}^n)$, the equality*

$$\begin{aligned} \|f\|_{H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)}^2 &= \frac{a_n}{4} \left[\|I^2 f\|_{H_{t+5/2}^{s+5/2}}^2 + 4\Im(I^2 f, Z(I^1 f))_{H_{t+2}^{s+2}} + 2 \sum_i \|\Xi_i(I^1 f)\|_{H_{t+3/2}^{s+3/2}}^2 \right. \\ &\quad + 2\|I^1 f\|_{H_{t+3/2}^{s+3/2}}^2 + 2\|Z(I^1 f)\|_{H_{t+3/2}^{s+3/2}}^2 + 2\Re(I^2 f, W(I^0 f))_{H_{t+2}^{s+2}} \\ &\quad - 4\Re(I^2 f, I^0 f)_{H_{t+3/2}^{s+3/2}} - 2\Re(I^2 f, Q(I^0 f))_{H_{t+3/2}^{s+3/2}} \\ &\quad - 4\Re(\Xi^i(I^1 f), x_i I^0 f)_{H_{t+3/2}^{s+3/2}} + 8\Im(Z(I^1 f), I^0 f)_{H_{t+1}^{s+1}} \\ &\quad + 4\Im(I^1 f, Z(I^0 f))_{H_{t+1}^{s+1}} + 4\Im(\Xi^i(I^1 f), Z\Xi_i(I^0 f))_{H_{t+1}^{s+1}} \\ &\quad + \sum_i \|x_i I^0 f\|_{H_{t+3/2}^{s+3/2}}^2 - 2\Im(x^i I^0 f, Z\Xi_i(I^0 f))_{H_{t+1}^{s+1}} \\ &\quad - 4\Re(W(I^0 f), I^0 f)_{H_{t+1}^{s+1}} + \sum_{i,j} \|\Xi_i \Xi_j(I^0 f)\|_{H_{t+1/2}^{s+1/2}}^2 \\ &\quad \left. + 4\Re(\Xi^i \Xi_i(I^0 f), I^0 f)_{H_{t+1/2}^{s+1/2}} + \sum_i \|\Xi_i(I^0 f)\|_{H_{t+1/2}^{s+1/2}}^2 + 4n\|I^0 f\|_{H_{t+1/2}^{s+1/2}}^2 \right] \end{aligned} \quad (4.43)$$

holds for every $s \in \mathbb{R}$ and every $t > -n/2$, where the constant a_n is defined by (2.13).

As before, the abbreviated notation $\|\cdot\|_{H_{i'}^{s'}}$ for the norm $\|\cdot\|_{H_{i'}^{s'}(T\mathbb{S}^{n-1})}$ is used on the right-hand side of (4.43)

Sketch of the proof. The proof follows the same line as the proof of Theorem 4.5 although all calculations are more cumbersome. We will present key formulas only.

The Reshetnyak formula (2.12) for scalar functions implies

$$\|f\|_{H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)}^2 = a_n \sum_{i,j} \|I f_{ij}\|_{H_{t+1/2}^{s+1/2}(T\mathbb{S}^{n-1})}^2 = \frac{a_n}{2\pi} \sum_{i,j} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1+|y|^2)^{s-t} |\widehat{I f_{ij}}(y, \xi)|^2 dy d\xi. \quad (4.44)$$

Applying the Fourier transform to equation (4.14) and using Lemma 4.4, we obtain the following analog of formula (4.34):

$$\widehat{I f_{ij}} = \frac{1}{2} \sigma(ij) \left(-y_i y_j \widehat{I^2 f} - 2i y_i \Xi_j \widehat{I^1 f} - 2i \xi_i y_j \widehat{I^1 f} + \Xi_i \Xi_j \widehat{I^0 f} - y_i X_j \widehat{I^0 f} + 3\xi_i \Xi_j \widehat{I^0 f} + 2\delta_{ij} \widehat{I^0 f} \right).$$

This implies

$$\left| \sum_{i,j} \widehat{I}f_{ij}(y, \xi) \right|^2 = \frac{1}{4} \sum_{i,j=1}^n \left| \sigma(ij) \left(-y_i y_j \widehat{I}^2 f(y, \xi) - 2i y_i (\Xi_j \widehat{I}^1 f)(y, \xi) - 2i \xi_i y_j \widehat{I}^1 f(y, \xi) \right. \right. \\ \left. \left. + (\Xi_i \Xi_j \widehat{I}^0 f)(y, \xi) - y_i (X_j \widehat{I}^0 f)(y, \xi) + 3\xi_i (\Xi_j \widehat{I}^0 f)(y, \xi) + 2\delta_{ij} \widehat{I}^0 f(y, \xi) \right) \right|^2.$$

Substituting this expression into (4.44), we get,

$$\|f\|_{H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)}^2 = \frac{a_n}{4} \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} \\ \times \sum_{i,j=1}^n \left| \sigma(ij) \left(-y_i y_j \widehat{I}^2 f(y, \xi) - 2i y_i (\Xi_j \widehat{I}^1 f)(y, \xi) - 2i \xi_i y_j \widehat{I}^1 f(y, \xi) \right. \right. \\ \left. \left. + (\Xi_i \Xi_j \widehat{I}^0 f)(y, \xi) - y_i (X_j \widehat{I}^0 f)(y, \xi) + 3\xi_i (\Xi_j \widehat{I}^0 f)(y, \xi) + 2\delta_{ij} \widehat{I}^0 f(y, \xi) \right) \right|^2 dy d\xi. \quad (4.45)$$

By pure algebraic calculations, on using the identities $|\xi|^2 = 1$ and $\langle \xi, y \rangle = 0$, combined with Lemma 4.1, we obtain the following analog of (4.36):

$$\sum_{i,j=1}^n \left| \sigma(ij) \left(-y_i y_j \widehat{I}^2 f(y, \xi) - 2i y_i (\Xi_j \widehat{I}^1 f)(y, \xi) - 2i \xi_i y_j \widehat{I}^1 f(y, \xi) \right. \right. \\ \left. \left. + (\Xi_i \Xi_j \widehat{I}^0 f)(y, \xi) - y_i (X_j \widehat{I}^0 f)(y, \xi) + 3\xi_i (\Xi_j \widehat{I}^0 f)(y, \xi) + 2\delta_{ij} \widehat{I}^0 f(y, \xi) \right) \right|^2 \\ = |y|^4 |\widehat{I}^2 f|^2 + 2|y|^2 \sum_i |\Xi_i \widehat{I}^1 f|^2 + 2|y|^2 |\widehat{I}^1 f|^2 + 2|y^i \Xi_i \widehat{I}^1 f|^2 \\ + \sum_{i,j} |\Xi_i \Xi_j \widehat{I}^0 f|^2 + |y|^2 \sum_i |X_i \widehat{I}^0 f|^2 + \sum_i |\Xi_i \widehat{I}^0 f|^2 + 4n |\widehat{I}^0 f|^2 \\ + 4|y|^2 \Im(\widehat{I}^2 f \cdot y^i \Xi_i \widehat{I}^1 f) - 2\Re(\widehat{I}^2 f \cdot y^i y^j \Xi_i \Xi_j \widehat{I}^0 f) \\ + 2|y|^2 \Re(\widehat{I}^2 f \cdot y^i X_i \widehat{I}^0 f) - 4|y|^2 \Re(\widehat{I}^2 f \cdot \widehat{I}^0 f) \\ + 4\Im(\Xi^i \widehat{I}^1 f \cdot y^j \Xi_j \Xi_i \widehat{I}^0 f) - 4|y|^2 \Im(\Xi^i \widehat{I}^1 f \cdot X_i \widehat{I}^0 f) \\ + 8\Im(y^i \Xi_i \widehat{I}^1 f \cdot \widehat{I}^0 f) + 4\Im(\widehat{I}^1 f \cdot y^i \Xi_i \widehat{I}^0 f) \\ - 2\Re(X^i \widehat{I}^0 f \cdot y^j \Xi_j \Xi_i \widehat{I}^0 f) + 4\Re(\Xi^i \Xi_i \widehat{I}^0 f \cdot \widehat{I}^0 f) - 4\Re(y^i X_i \widehat{I}^0 f \cdot \widehat{I}^0 f).$$

Substitute this value into (4.45)

$$\begin{aligned}
\frac{4}{a_n} \|f\|_{H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)}^2 &= \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} \left[|y|^4 |\widehat{I^2 f}(y, \xi)|^2 + 2|y|^2 |\widehat{I^1 f}(y, \xi)|^2 \right. \\
&+ 4n |\widehat{I^0 f}(y, \xi)|^2 - 4|y|^2 \Re(\widehat{I^2 f}(y, \xi) \overline{\widehat{I^0 f}(y, \xi)}) \\
&+ 2|y|^2 \sum_i |(\Xi_i \widehat{I^1 f})(y, \xi)|^2 + \sum_i |(\Xi_i \widehat{I^0 f})(y, \xi)|^2 \\
&+ \sum_{i,j} |(\Xi_i \Xi_j \widehat{I^0 f})(y, \xi)|^2 + 2|(y^i \Xi_i \widehat{I^1 f})(y, \xi)|^2 + |y|^2 \sum_i |(X_i \widehat{I^0 f})(y, \xi)|^2 \\
&+ 4|y|^2 \Im(\widehat{I^2 f}(y, \xi) \overline{(y^i \Xi_i \widehat{I^1 f})(y, \xi)}) - 2\Re(\widehat{I^2 f}(y, \xi) \overline{(y^i y^j \Xi_i \Xi_j \widehat{I^0 f})(y, \xi)}) \\
&+ 2|y|^2 \Re(\widehat{I^2 f}(y, \xi) \overline{(y^i X_i \widehat{I^0 f})(y, \xi)}) + 4\Im((\Xi^i \widehat{I^1 f})(y, \xi) \overline{(y^j \Xi_j \Xi_i \widehat{I^0 f})(y, \xi)}) \\
&- 4|y|^2 \Im((\Xi^i \widehat{I^1 f})(y, \xi) \overline{(X_i \widehat{I^0 f})(y, \xi)}) + 8\Im((y^i \Xi_i \widehat{I^1 f})(y, \xi) \overline{\widehat{I^0 f}(y, \xi)}) \\
&+ 4\Im(\widehat{I^1 f}(y, \xi) \overline{(y^i \Xi_i \widehat{I^0 f})(y, \xi)}) - 2\Re((X^i \widehat{I^0 f})(y, \xi) \overline{(y^j \Xi_j \Xi_i \widehat{I^0 f})(y, \xi)}) \\
&\left. + 4\Re((\Xi^i \Xi_i \widehat{I^0 f})(y, \xi) \overline{\widehat{I^0 f}(y, \xi)}) - 4\Re((y^i X_i \widehat{I^0 f})(y, \xi) \overline{\widehat{I^0 f}(y, \xi)}) \right] dy d\xi.
\end{aligned} \tag{4.46}$$

There are 19 terms in the brackets on the right-hand side of (4.46) which are in one-to-one correspondence with terms on the right-hand side of (4.43). On using Lemma 4.4 and operators Z, Q, W , one checks that the integral of each term in the brackets on the right-hand side of (4.46) (with the weight written before the brackets) is equal to the corresponding term on the right-hand side of (4.43). \square

Repeating arguments of Section 4.2, we introduce the following norm on the space $\mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1})$:

$$\begin{aligned}
\|(\varphi, \psi, \chi)\|_{\mathcal{H}_t^{2,s}(T\mathbb{S}^{n-1})}^2 &= \frac{a_n}{4} \left[\|\chi\|_{H_{t+5/2}^{s+5/2}}^2 + 4\Im(\chi, Z\psi)_{H_{t+2}^{s+2}} + 2 \sum_i \|\Xi_i \psi\|_{H_{t+3/2}^{s+3/2}}^2 \right. \\
&+ 2\|\psi\|_{H_{t+3/2}^{s+3/2}}^2 + 2\|Z\psi\|_{H_{t+3/2}^{s+3/2}}^2 + 2\Re(\chi, W\varphi)_{H_{t+2}^{s+2}} \\
&- 4\Re(\chi, \varphi)_{H_{t+3/2}^{s+3/2}} - 2\Re(\chi, Q\varphi)_{H_{t+3/2}^{s+3/2}} \\
&- 4\Re(\Xi^i \psi, x_i \varphi)_{H_{t+3/2}^{s+3/2}} + 8\Im(Z\psi, \varphi)_{H_{t+1}^{s+1}} \\
&+ 4\Im(\psi, Z\varphi)_{H_{t+1}^{s+1}} + 4\Im(\Xi^i \psi, Z\Xi_i \varphi)_{H_{t+1}^{s+1}} \\
&+ \sum_i \|x_i \varphi\|_{H_{t+3/2}^{s+3/2}}^2 - 2\Im(x^i \varphi, Z\Xi_i \varphi)_{H_{t+1}^{s+1}} \\
&- 4\Re(W\varphi, \varphi)_{H_{t+1}^{s+1}} + \sum_{i,j} \|\Xi_i \Xi_j \varphi\|_{H_{t+1/2}^{s+1/2}}^2 \\
&\left. + 4\Re(\Xi^i \Xi_i \varphi, \varphi)_{H_{t+1/2}^{s+1/2}} + \sum_i \|\Xi_i \varphi\|_{H_{t+1/2}^{s+1/2}}^2 + 4n \|\varphi\|_{H_{t+1/2}^{s+1/2}}^2 \right].
\end{aligned} \tag{4.47}$$

Lemma 4.9. *The right-hand side of (4.47) is positive unless $(\varphi, \psi, \chi) = (0, 0, 0)$.*

This statement was actually proved before. Indeed, the integral

$$A[\varphi, \psi, \chi] = \frac{a_n}{4} \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{\xi^\perp} |y|^{2t+1} (1 + |y|^2)^{s-t} \sum_{i,j=1}^n \left| \sigma(ij) \left(-y_i y_j \widehat{\chi}(y, \xi) - 2i y_i (\Xi_j \widehat{\psi})(y, \xi) \right. \right. \\ \left. \left. - 2i \xi_i y_j \widehat{\psi}(y, \xi) + (\Xi_i \Xi_j \widehat{\varphi})(y, \xi) - y_i (X_j \widehat{\varphi})(y, \xi) + 3\xi_i (\Xi_j \widehat{\varphi})(y, \xi) + 2\delta_{ij} \widehat{\varphi}(y, \xi) \right) \right|^2 dy d\xi \quad (4.48)$$

is positive unless $\varphi = \psi = \chi = 0$. The right-hand side of (4.45) is equal to $A[I^0 f, I^1 f, I^2 f]$. After formula (4.45), the proof of Theorem 4.8 consists of some transformations of the right-hand side of (4.45). No specific property of the functions $(I^0 f, I^1 f, I^2 f)$ is used in the transformations, i.e., the right-hand side of (4.48) can be transformed in the same manner. In this way, we demonstrate that $A[\varphi, \psi, \chi]$ is equal to the right-hand side of (4.47).

We define the Hilbert space $\mathcal{H}_t^{2,s}(T\mathbb{S}^{n-1})$ as the completion of $\mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1})$ with respect to the norm (4.47).

Corollary 4.10. *The map*

$$\mathcal{S}(\mathbb{R}^n; S^2\mathbb{R}^n) \rightarrow \mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1}) \oplus \mathcal{S}(T\mathbb{S}^{n-1}), \quad f \mapsto (I^0 f, I^1 f, I^2 f)$$

extends to the linear isometric embedding

$$H_t^s(\mathbb{R}^n; \mathbb{C}^n) \rightarrow \mathcal{H}_t^{2,s}(T\mathbb{S}^{n-1})$$

for every $s \in \mathbb{R}$ and every $t > -n/2$. In other words, the equality

$$\|f\|_{H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)} = \|(I^0 f, I^1 f, I^2 f)\|_{\mathcal{H}_t^{2,s}(T\mathbb{S}^{n-1})}$$

holds for all $f \in H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)$.

Estimating scalar products on the right-hand side of (4.43) with the help of (4.38), we also obtain

Corollary 4.11. *For every $s \in \mathbb{R}$ and for every $t > -n/2$, the stability estimate*

$$\|f\|_{H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)}^2 \leq b_n \left[\|I^2 f\|_{H_{t+5/2}^{s+5/2}}^2 + \sum_i \|\Xi_i(I^1 f)\|_{H_{t+3/2}^{s+3/2}}^2 + \|I^1 f\|_{H_{t+3/2}^{s+3/2}}^2 + \|Z(I^1 f)\|_{H_{t+3/2}^{s+3/2}}^2 \right. \\ \left. + \sum_i \|x_i I^0 f\|_{H_{t+3/2}^{s+3/2}}^2 + \sum_{i,j} \|\Xi_i \Xi_j(I^0 f)\|_{H_{t+1/2}^{s+1/2}}^2 + \|I^0 f\|_{H_{t+1/2}^{s+1/2}}^2 + \sum_{i=1}^n \|\Xi_i(I^0 f)\|_{H_{t+1/2}^{s+1/2}}^2 \right] \quad (4.49)$$

holds for all $f \in H_t^s(\mathbb{R}^n; S^2\mathbb{R}^n)$ with some constant b_n depending on n only.

In principle, our approach works for every m , i.e., the Reshetnyak formula can be obtained which expresses the norm $\|f\|_{H_t^s(\mathbb{R}^n; S^m\mathbb{R}^n)}$ through some norm of $(I^0 f, \dots, I^m f)$ for a rank m symmetric tensor field f . But the length of the formula grows fast with m . For example, the Reshetnyak formula contains more than 250 terms in the case of $m = 3$. Unfortunately, we are not able to bring the formula into a compact form.

5. THE TWO-DIMENSIONAL CASE

Due to the existence of natural coordinates on $TS^1 = \mathbb{S}^1 \times \mathbb{R}$, all our formulas are simplified in the 2D-case. In particular, we do not need to use the operators X_i, Ξ_i as well as the operators Z, Q, W participating in (4.43). Actually our work on the subject was started with considering the 2D-case.

The coordinates (p, θ) on the two-dimensional manifold TS^1 are defined by

$$x = p(-\sin \theta, \cos \theta), \quad \xi = (\cos \theta, \sin \theta) \quad \text{for } (x, \xi) \in TS^1.$$

The operators X_i and Ξ_i are easily expressed through $\frac{\partial}{\partial p}$ and $\frac{\partial}{\partial \theta}$

$$X_1 = -\sin \theta \frac{\partial}{\partial p}, \quad X_2 = \cos \theta \frac{\partial}{\partial p}, \quad \Xi_1 = -\sin \theta \frac{\partial}{\partial \theta}, \quad \Xi_2 = \cos \theta \frac{\partial}{\partial \theta}. \quad (5.1)$$

In particular, the first term on the right-hand side of (4.37) can be written as

$$\sum_{i=1}^2 \|\Xi_i(I^0 f)\|_{H_{t+1/2}^{s+1/2}(TS^1)}^2 = \left\| \frac{\partial(I^0 f)}{\partial \theta} \right\|_{H_{t+1/2}^{s+1/2}(TS^1)}^2. \quad (5.2)$$

The Fourier transform $\mathcal{S}(TS^1) \rightarrow \mathcal{S}(TS^1)$, $\varphi \mapsto \hat{\varphi}$ is just the one-dimensional Fourier transform in the variable p with the Fourier dual variable q , where θ stands as a parameter

$$\hat{\varphi}(q, \theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqp} \varphi(p, \theta) dp.$$

The *Hilbert transform* $H : H_t^s(TS^1) \rightarrow H_t^s(TS^1)$ is defined by

$$\widehat{H\varphi}(q, \theta) = \text{sgn}(q) \hat{\varphi}(q, \theta).$$

As is seen from (4.33) and (5.1),

$$Z = H \frac{\partial}{\partial \theta}. \quad (5.3)$$

With the help of (5.2) and (5.3), the Reshetnyak formula (4.32) for vector fields takes the following form in the two-dimensional case:

$$\begin{aligned} \|f\|_{H_t^s(\mathbb{R}^2; \mathbb{C}^2)}^2 &= \frac{1}{2} \left[\|\partial_\theta(I^0 f)\|_{H_{t+1/2}^{s+1/2}(TS^1)}^2 + \|I^0 f\|_{H_{t+1/2}^{s+1/2}(TS^1)}^2 \right. \\ &\quad \left. + \|I^1 f\|_{H_{t+3/2}^{s+3/2}(TS^1)}^2 + 2\Re(iH\partial_\theta(I^0 f), I^1 f)_{H_{t+1}^{s+1}(TS^1)} \right], \end{aligned}$$

and the stability estimate (4.42) takes the form

$$\|f\|_{H_t^s(\mathbb{R}^2; \mathbb{C}^2)}^2 \leq \|\partial_\theta(I^0 f)\|_{H_{t+1/2}^{s+1/2}(TS^1)}^2 + \|I^0 f\|_{H_{t+1/2}^{s+1/2}(TS^1)}^2 + \|I^1 f\|_{H_{t+3/2}^{s+3/2}(TS^1)}^2.$$

Formulas (4.43) and (4.49) admit similar simplifications in the 2D-case. Omitting details, we present the result. the Reshetnyak formula (4.43) for second rank tensor fields

takes the following form in the two-dimensional case:

$$\begin{aligned}
8\|f\|_{H_t^s(\mathbb{R}^2; S^2\mathbb{R}^2)}^2 &= \|I^2f\|_{H_{t+5/2}^{s+5/2}}^2 + 4\|\partial_\theta(I^1f)\|_{H_{t+3/2}^{s+3/2}}^2 + 2\|I^1f\|_{H_{t+3/2}^{s+3/2}}^2 \\
&+ \|\partial_\theta^2(I^0f)\|_{H_{t+1/2}^{s+1/2}}^2 - 2\|\partial_\theta(I^0f)\|_{H_{t+1/2}^{s+1/2}}^2 + \|pI^0f\|_{H_{t+3/2}^{s+3/2}}^2 + 8\|I^0f\|_{H_{t+1/2}^{s+1/2}}^2 \\
&- 4\Re(iH(I^2f), \partial_\theta(I^1f))_{H_{t+2}^{s+2}} + 2\Re(iH(I^2f), pI^0f)_{H_{t+2}^{s+2}} - 2\Re(I^2f, \partial_\theta^2(I^0f))_{H_{t+3/2}^{s+3/2}} \\
&- 4\Re(I^2f, I^0f)_{H_{t+3/2}^{s+3/2}} - 4\Re(\partial_\theta(I^1f), pI^0f)_{H_{t+3/2}^{s+3/2}} - 4\Re(iH\partial_\theta(I^1f), \partial_\theta^2(I^0f))_{H_{t+1}^{s+1}} \\
&+ 4\Re(iH(I^1f), \partial_\theta(I^0f))_{H_{t+1}^{s+1}} - 2\Re(iH\partial_\theta^2(I^0f), pI^0f)_{H_{t+1}^{s+1}} + 4\Re(iH(pI^0f), I^0f)_{H_{t+1}^{s+1}}.
\end{aligned}$$

Here the shorter notation $H_t^{s'}$ is used instead of $H_t^{s'}(TS^1)$ on the right-hand side. The corresponding stability estimate is

$$\begin{aligned}
\|f\|_{H_t^s(\mathbb{R}^2; S^2\mathbb{R}^2)}^2 &\leq 6 \left[\|I^2f\|_{H_{t+5/2}^{s+5/2}}^2 + \|\partial_\theta(I^1f)\|_{H_{t+3/2}^{s+3/2}}^2 + \|I^1f\|_{H_{t+3/2}^{s+3/2}}^2 \right. \\
&\quad \left. + \|pI^0f\|_{H_{t+3/2}^{s+3/2}}^2 + \|\partial_\theta^2(I^0f)\|_{H_{t+1/2}^{s+1/2}}^2 + \|\partial_\theta(I^0f)\|_{H_{t+1/2}^{s+1/2}}^2 + \|I^0f\|_{H_{t+1/2}^{s+1/2}}^2 \right].
\end{aligned}$$

Even in the 2D-case, the Reshetnyak formula for rank 3 symmetric tensor fields is too long to be presented here.

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E-mail address: vkrishnan@tifrbng.res.in, ramesh@tifrbng.res.in, suman@math.tifrbng.res.in, sharaf@math.nsc.ru

*CORRESPONDING AUTHOR

†TIFR CENTRE FOR APPLICABLE MATHEMATICS, SHARADA NAGAR, CHIKKABOMMASANDRA, YELAHANKA NEW TOWN, BANGALORE, INDIA

‡SOBOLEV INSTITUTE OF MATHEMATICS; 4 KOPTYUG AVENUE, NOVOSIBIRSK, 630090, RUSSIA; NOVOSIBIRSK STATE UNIVERSITY, 2 PIROGOV STREET, 630090, RUSSIA