POINT SOURCES AND STABILITY FOR AN INVERSE PROBLEM FOR A HYPERBOLIC PDE WITH SPACE AND TIME DEPENDENT COEFFICIENTS

VENKATESWARAN P. KRISHNAN*, RAKESH†, AND SOUMEN SENAPATI‡

Abstract. We study stability aspects for the determination of space and time-dependent lower order perturbations of the wave operator in three space dimensions with point sources. The problems under consideration here are formally determined and we establish Lipschitz stability results for these problems. The main tool in our analysis is a modified version of Bukgheĭm-Klibanov method based on Carleman estimates.

Keywords: Formally determined hyperbolic inverse problem, stability, time dependent coefficients, Carleman estimates.

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1. Introduction and main results

Throughout this article, all functions are real valued, T denotes a positive real number, B denotes the origin centered open unit ball in $\mathbb{R}^d$ for any positive integer d and, for $\rho > 0$, ρB is the origin centered open ball of radius ρ in $\mathbb{R}^d$.

For functions $a(x,t), c(x,t)$ and the vector field $b(x,t) = (b^1(x,t), b^2(x,t), b^3(x,t))$ on $\mathbb{R}^3 \times \mathbb{R}$, define the hyperbolic operator

$$\mathcal{L}_{a,b,c} := (\partial_t - a)^2 - (\nabla - b)^2 + c = \Box - 2a\partial_t + 2b\cdot\nabla + q \tag{1.1}$$

where

$$q = c - a_t + \nabla\cdot b + a^2 - |b|^2.$$

To avoid introducing too many symbols, we use $\mathcal{L}_{a,b,c}$ and $\mathcal{L}_{a,b,q}$ interchangeably since the form of the operator will be clear from the context.

Suppose $a(x,t), c(x,t)$ and $b(x,t)$ are smooth compactly supported functions and a vector field on $\mathbb{R}^3\times\mathbb{R}$ with support in $\overline{B}\times\mathbb{R}$. Given $\xi \in \mathbb{R}^3 \setminus \overline{B}$, $\tau\in\mathbb{R}$, let $U(x,t;\xi,\tau)$ be the solution of the IVP

$$\mathcal{L}_{a,b,c}U(x,t;\xi,\tau) = 4\pi H(t-\tau)\delta(x-\xi), \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{1.2}$$

$$U(x,t;\xi,\tau) = 0, \quad \text{for } x\in\mathbb{R}^3,\ t<0, \tag{1.3}$$

and let $V(x,t;\xi,\tau)$ be the solution of the IVP

$$\mathcal{L}_{a,b,c}V(x,t;\xi,\tau) = 4\pi\delta(x-\xi,t-\tau), \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{1.4}$$

$$V(x,t;\xi,\tau) = 0, \quad \text{for } x\in\mathbb{R}^3,\ t<0. \tag{1.5}$$

Define the forward map

$$\mathcal{F} : (a,b,c) \to [U,U_t,V,V_t](x,T;\xi,\tau)|_{x\in\mathbb{R}^3,\xi\in E,\tau\in(-\infty,T]} \tag{1.6}$$

which measures the medium response at the final time $t = T$, to waves generated by a point source at ξ in a **finite** subset E of $\mathbb{R}^3$, with sources activated at times τ varying over the interval $(-\infty,T]$. Here (a,b,c) represents the medium properties and the medium is uniform outside the cylinder $\overline{B}\times[0,T]$. Our goal is to study the injectivity and stability of $\mathcal{F}$. The problem is formally determined in the sense that the data set depends on four real parameters - three for the receiver locations $(x, t{=}T) \in \mathbb{R}^3\times\{t{=}T\}$ and one for the time delay τ - while the unknown coefficients (a,b,c) are also functions of the four variables $(x,t)\in\mathbb{R}^3\times\mathbb{R}$.

A point source inverse back-scattering problem in $\mathbb{R}^3$ involving the recovery of time-independent potential by measuring the response at a point due to a source located at the same point with their locations varying over the surface of a sphere was considered in [7]. They showed the unique recovery of angularly controlled potentials, in particular, radial potentials, from such formally determined data. This was further investigated in [1], where a logarithmic stability estimate for the recovery of time-independent angularly controlled potentials for the point source inverse backscattering problem was shown.

In this paper, as already mentioned, we consider the inverse problem of determing time-dependent first order perturbations of the wave operator from data measured using point sources. This is a follow-up of our previous work [4], in which we derived Lipschitz stability estimates for the determination of the coefficients a, b (up to a gauge term) and c in (1.1) in space dimensions $n\geq 2$ and with plane wave sources. However, in the current work with point sources, similar to [7, 1], we limit ourselves to space dimensions $n = 3$ since the ansatz involving the fundamental solution of the wave equation for U and V above becomes unwieldy in higher dimensions.

Similar to the work [4], we derive uniqueness and Lipschitz stability estimates for the recovery of time-dependent coefficients a, b (up to a gauge term) and c for a formally determined inverse problem with point sources. Our proofs are based on suitable modifications of the ideas of Bukhgeĭm and Klibanov [2] which were based on Carleman estimates. Finally, we mention the related work [5] that deals with a recovery of time-independent first order coefficients of a hyperbolic PDE in a formally determined set-up as well.

A detailed survey of prior work in hyperbolic inverse problems for time-independent/time-dependent lower order coefficients with constant/variable coefficients in the principal term in which (a) data is measured only on the lateral boundary (or a part of it), (b) data is not measured on the top part corresponding to time $t = T$, (c) sources are not present at the intial time $t = 0$ and (d) data is in the form of far-field pattern in the frequency domain, is given in our earlier work [4]. For this reason, we do not repeat them here.

Before discussing the main results of the article, we introduce some definitions and notation. Given $\xi \in \mathbb{R}^3 \setminus \overline{B}$ and $\tau \in \mathbb{R}$, define the conical region

$$Q_{\xi,\tau} = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R};\ |x-\xi| + \tau \le t \le T\}$$

and denote its top (horizontal) and conical boundaries by

$$H_{\xi,\tau} = Q_{\xi,\tau} \cap \{t = T\}, \qquad C_{\xi,\tau} = Q_{\xi,\tau} \cap \{t = \tau + |x-\xi|\},$$

respectively.

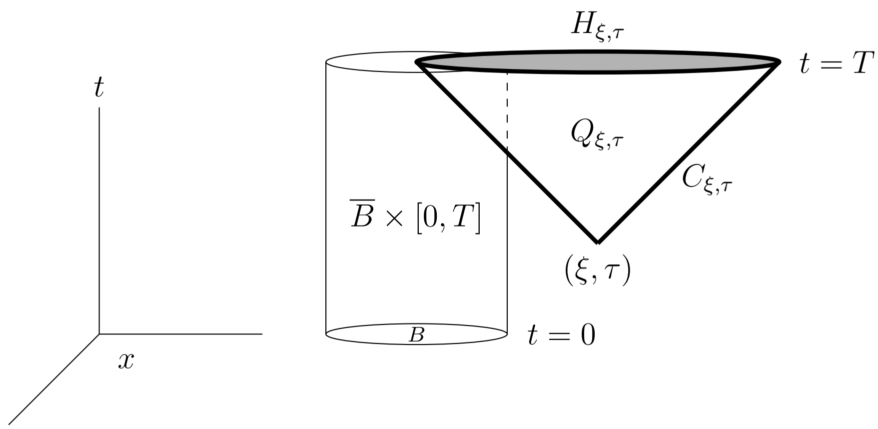

FIGURE 1. The conical domain $Q_{\xi,\tau}$ and its boundaries

Given $\sigma > 0$, M a submanifold of $\mathbb{R}^3 \times \mathbb{R}$, and a function $f : M \to \mathbb{R}$, define the weighted norms

$$\|f\|_{0,M,\sigma} = \left(\int_M e^{2\sigma t}|f|^2\right)^{1/2}, \quad \|f\|_{1,M,\sigma} = \left(\int_M e^{2\sigma t}\left(|\nabla_M f|^2 + \sigma^2|f|^2\right)\right)^{1/2}$$

where ∇_M consists of the first order derivatives in directions tangential to M. For $x, \xi \in \mathbb{R}^3$, $x \ne \xi$, define

$$r = |x-\xi|,\ \theta = \frac{x-\xi}{|x-\xi|}, \quad \partial_r = \theta \cdot \nabla. \tag{1.7}$$

For a compactly supported smooth function a and vector field b on $\mathbb{R}^3 \times \mathbb{R}$, and $\xi \in \mathbb{R}^3$ such that $\{\xi\} \times \mathbb{R}$ is disjoint from the supports of a, b, define

$$\alpha(x,t;\xi) = \frac{1}{r}\exp\left(\int_0^r (a + \theta \cdot b)(x - s\theta, t-s)\ \mathrm{d}s\right), \quad x \ne \xi. \tag{1.8}$$

Note that $\alpha(x,t;\xi) = r^{-1}$ in a punctured cylindrical neighborhood of $\{\xi\}\times\mathbb{R}$ and α satisfies the equivalent transport equations

$$\left(\partial_t + \theta\cdot\nabla + r^{-1}\right)\alpha = (a+\theta\cdot b)\alpha, \qquad (\partial_t + \partial_r - (a+\theta\cdot b))(r\alpha) = 0, \qquad x \neq \xi. \tag{1.9}$$

This follows from the identity

$$r\left(\partial_t + \theta\cdot\nabla + r^{-1}\right)\alpha = (\partial_t + \partial_r)(r\alpha)$$

and that

$$\begin{aligned}
&(r\alpha)^{-1}(\partial_t+\partial_r)(r\alpha)\\
&= \exp\left(-\int_0^r (a+\theta\cdot b)(x-s\theta,t-s)\,\mathrm{d}s\right)(\partial_t+\partial_r)\exp\left(\int_0^r (a+\theta\cdot b)(x-s\theta,t-s)\,\mathrm{d}s\right)\\
&= \int_0^r (a_t+\theta\cdot b_t + a_r + \theta\cdot b_r)(x-s\theta,t-s)\,\mathrm{d}s + (a+\theta\cdot b)(x-r\theta,t-r)\\
&= -\int_0^r \frac{\mathrm{d}}{\mathrm{d}s}(a+\theta\cdot b)(x-s\theta,t-s)\,\mathrm{d}s + (a+\theta\cdot b)(x-r\theta,t-r).\\
&= (a+\theta\cdot b)(x,t).
\end{aligned}$$

We also define the useful first order operators

$$\mathcal{M} = -2a\partial_t + 2b\cdot\nabla + q, \quad \mathcal{T} = \partial_t + \theta\cdot\nabla - (a+\theta\cdot b) + r^{-1}, \qquad x\neq\xi;$$

note that $\mathcal{M}$ is zero in a punctured cylindrical neighborhood of $\{\xi\}\times\mathbb{R}$ and (1.9) may be rewritten as

$$\mathcal{T}\alpha = 0, \qquad x\neq\xi. \tag{1.10}$$

We first address the structure of U, V and the well-posedness of the IVPs defining U, V.

Proposition 1.1. *If a, c, and b are compactly supported smooth functions and a vector field on $\mathbb{R}^3\times\mathbb{R}$, respectively and $\{\xi\}\times\mathbb{R}$ is disjoint from the support of a, b, c, then the IVP*

$$\mathcal{L}_{a,b,c}U(x,t;\xi,\tau) = 4\pi H(t-\tau)\delta(x-\xi), \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{1.11}$$

$$U(x,t;\xi,\tau) = 0, \quad \text{for } x\in\mathbb{R}^3,\ t<0, \tag{1.12}$$

admits a unique distributional solution $U(x,t;\xi,\tau)$. Further,

$$U(x,t;\xi,\tau) = \frac{H(t-\tau-|x-\xi|)}{|x-\xi|} + u(x,t;\xi,\tau)H(t-\tau-|x-\xi|),$$

where $u(x,t;\xi,\tau)$ is a smooth function in the region $\{(x,t)\in\mathbb{R}^3\times\mathbb{R};\ t\geq\tau+|x-\xi|\}$ and is a smooth solution of the characteristic BVP

$$\mathcal{L}_{a,b,c}u = -\mathcal{M}\left(|x-\xi|^{-1}\right), \quad t>\tau+|x-\xi|, \tag{1.13}$$

$$u(x,t;\xi,\tau) = \alpha(x,t;\xi) - |x-\xi|^{-1}, \quad t=\tau+|x-\xi|,\ x\neq\xi. \tag{1.14}$$

Finally, if the compactly supported coefficients a, b, c satisfy $\|[a,b,c]\|_{C^{20}(\mathbb{R}^3\times\mathbb{R})}\leq M$, then

$$\|u\|_{C^3(Q_{\xi,\tau})} \leq C$$

where C depends only on T, M and the reciprocal of the distance of $\{\xi\}\times\mathbb{R}$ from the support of a, b, c.

Proposition 1.2. *If a, c and b are compactly supported smooth functions and a vector field on $\mathbb{R}^3\times\mathbb{R}$, respectively, and $\{\xi\}\times\mathbb{R}$ is disjoint from the support of a, b, c, the IVP*

$$\mathcal{L}_{a,b,q}V(x,t;\xi,\tau) = 4\pi\delta(x-\xi, t-\tau), \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{1.15}$$

$$V(x,t;\xi,\tau) = 0, \quad \text{for } x\in\mathbb{R}^3,\ t<0. \tag{1.16}$$

admits a unique distributional solution $V(x,t;\xi,\tau)$ of the form

$$V(x,t;\xi,\tau) = \alpha(x,t;\xi)\delta(t-\tau-|x-\xi|) + v(x,t;\xi,\tau)H(t-\tau-|x-\xi|)$$

where $v(x,t;\xi,\tau)$ is a smooth function in the region $\{(x,t)\in\mathbb{R}^3\times\mathbb{R};\ t\geq\tau+|x-\xi|\}$ and solves the characteristic BVP

$$\mathcal{L}_{a,b,c}v = 0, \quad t>\tau+|x-\xi|, \tag{1.17}$$

$$\mathcal{T}v = -\frac{1}{2}\mathcal{L}_{a,b,c}\alpha, \quad t=\tau+|x-\xi|,\ x\neq\xi. \tag{1.18}$$

Finally, if the compactly supported coefficients a, b, c satisfy $\|[a,b,c]\|_{C^{22}(\mathbb{R}^3\times\mathbb{R})}\leq M$, then

$$\|v\|_{C^3(Q_{\xi,\tau})}\leq C$$

where C depends only on T, M and the reciprocal of the distance of $\{\xi\}\times\mathbb{R}$ from the support a, b, c

For future use we make several observations about u and v.

- Since a, b, c are supported away from $(x{=}\xi, t{=}\tau)$, in some neighborhood of $(x{=}\xi, t{=}\tau)$ we have

  $$\mathcal{M}=0,\ \alpha(x,t;\xi)=|x-\xi|^{-1},\ \mathcal{L}_{a,b,c}\alpha = -\Delta(|x-\xi|^{-1})=0, \qquad \text{for } x\neq\xi,$$
  $$U(x,t;\xi,\tau)=\frac{H(t-\tau-|x-\xi|)}{|x-\xi|}, \qquad V(x,t;\xi,\tau)=\frac{\delta(t-\tau-|x-\xi|)}{|x-\xi|}.$$

  Hence $u{=}0$ and $v{=}0$ in a neighborhood of $(x{=}\xi, t{=}\tau)$ and the singular terms $|x-\xi|^{-1}$ in (1.13), (1.14), (1.17), (1.18) will never be an issue.
- Suppose a, b, c are supported in $\overline{B}\times[0,T]$. We claim that for $\tau > T+1-|\xi|$ the values of u, v and their derivatives are zero on $t=T$. This is so because, for $\tau>T+1-|\xi|$, we have

  $$U(x,t;\xi,\tau)=\frac{H(t-\tau-|x-\xi|)}{|x-\xi|}, \qquad V(x,t;\xi,\tau)=\frac{\delta(t-\tau-|x-\xi|)}{|x-\xi|},$$

  which may be readily verified because $\mathcal{L}_{a,b,c}=\Box$ on the supports of the right hand sides of the two expressions.

- Suppose a, b, c are supported in $\overline{B} \times [0, T]$. If $\tau_1 < \tau_2 < -(1 + |\xi|)$ then the values of $[u, v](\cdot, \cdot, \xi, \tau_1)$ and $[u, v](\cdot, \cdot, \xi, \tau_2)$ and their derivatives on $t = T$ are the same. This is so because, for $\tau_1 < \tau_2 < -(1 + |\xi|)$, we have
$$U(x, t; \xi, \tau_1) - U(x, t; \xi, \tau_2) = \frac{H(t - \tau_1 - |x - \xi|)}{|x - \xi|} - \frac{H(t - \tau_2 - |x - \xi|)}{|x - \xi|}$$
$$V(x, t; \xi, \tau_1) - V(x, t; \xi, \tau_2) = \frac{\delta(t - \tau_1 - |x - \xi|)}{|x - \xi|} - \frac{\delta(t - \tau_2 - |x - \xi|)}{|x - \xi|};$$
this may be readily verified because $\mathcal{L}_{a,b,c} = \Box$ on the supports of the right hand sides of the two expressions.
- The previous two observations show that there is no new information about a, b, c in the values of u, v and their derivatives on $t = T$ for τ outside the interval $[-(1 + \xi|), T + 1 - |\xi|]$.

Now we describe the main results in our article. Our first result is about the stability for the problem of recovering q from the data generated by a point source at a fixed location in space but activated at different times τ.

Theorem 1.3 (Stability for q)**.** *Let a, b be a smooth function and a smooth vector field on $\mathbb{R}^3 \times \mathbb{R}$ with support in $\overline{B} \times [0, T]$ and $\xi \in \mathbb{R}^3 \setminus \overline{B}$. Given $M > 0$, for all smooth functions $q, \acute{q}$ on $\mathbb{R}^3 \times \mathbb{R}$ with support in $\overline{B} \times [0, T]$ and $\|[q, \acute{q}, a, b]\|_{C^{21}(\overline{B} \times [0,T])} \leq M$, we have*
$$\|q - \acute{q}\|_{0,\mathbb{R}^3 \times [0,T]} \preccurlyeq \int_{-1-|\xi|}^{T+1-|\xi|} \left(\|(v - \acute{v})(\cdot, T; \xi, \tau)\|_{1,H_{\xi,\tau}} + \|(v_t - \acute{v}_t)(\cdot, T; \xi, \tau)\|_{0,H_{\xi,\tau}} \right) \mathrm{d}\tau.$$
Here the constant is independent of $q, \acute{q}$, and v, $\acute{v}$ are the functions associated to (a, b, q) and $(a, b, \acute{q})$ guaranteed by Proposition 1.2.

The rest of our results pertain to the recovery of the vector field b and perhaps the functions a, c. For such results, we need sources at 4 locations diverse enough to generate data to separate a, b.

Definition 1.4. *Suppose d is a positive integer and D is a non-empty bounded open subset of $\mathbb{R}^d$. A set of locations $\xi_1, \cdots, \xi_{d+1}$ in $\mathbb{R}^d \setminus \overline{D}$ is said to be* **diverse** *with respect to D if*
$$\|[a, b]\| \leq C \|[a + \theta_1(x) \cdot b, \cdots, a + \theta_{d+1}(x) \cdot b]\|, \qquad \forall x \in \overline{D},\ \forall a \in \mathbb{R},\ \forall b \in \mathbb{R}^d, \tag{1.19}$$
for some constant C independent of a, b, x. Here $\|\cdot\|$ is the l^2 vector norm in $\mathbb{R}^{d+1}$ and
$$\theta_i(x) = \frac{x - \xi_i}{|x - \xi_i|}, \qquad x \in \overline{D}.$$

We do not have a characterization of all possible sets of locations diverse with respect to D but Proposition 8.1 gives two ways to construct many such sets. A consequence of Proposition 8.1 (see the remark after Proposition 8.1) is that if $\rho > 0$ then $Ne_1, Ne_2, Ne_3, N(e_1+e_2+e_3)/3$ is a diverse set of locations with respect to ρB if $N > \rho\sqrt{3}$. Here e_1, e_2, e_3 are the standard basis vectors in $\mathbb{R}^3$.

Our next result addresses the recovery of a, b when q is known.

Theorem 1.5 (Stability for a, b)**.** *Suppose q is a compactly supported smooth function in $\mathbb{R}^3 \times \mathbb{R}$ with support in $\overline{B} \times [0,T]$, and $\xi_1, \cdots, \xi_4$ is a diverse set of locations with respect to B. Given $M > 0$, if $a, \acute{a}, b, \acute{b}$ are smooth functions and vector fields on $\mathbb{R}^3 \times \mathbb{R}$ with support in $\overline{B} \times [0,T]$ and $\|[a, b, \acute{a}, \acute{b}, q]\|_{C^{19}(\overline{B}\times[0,T])} \le M$, we have*

$$\|[a - \acute{a}, b - \acute{b}]\|_{0,\mathbb{R}^3\times[0,T]} \preccurlyeq \sum_{i=1}^{4} \int_{-1-|\xi_i|}^{T+1-|\xi_i|} \|(u - \acute{u})(\cdot, T; \xi_i, \tau)\|_{1,H_{\xi_i,\tau}} + \|(u_t - \acute{u}_t)(\cdot, T; \xi_i, \tau)\|_{0,H_{\xi_i,\tau}} \,\mathrm{d}\tau.$$

Here the constant is independent of $a, b, \acute{a}, \acute{b}$, and u, $\acute{u}$ are the functions associated to (a, b, q) and $(\acute{a}, \acute{b}, q)$ guaranteed by Proposition 1.1.

Our next result addresses the uniqueness in the recovery of (a, b, c). As shown earlier, one expects to recover only $\operatorname{curl}(a, b)$ and c. Unfortunately, to obtain this result we need to restrict a, b to those for which $a + \theta_4 \cdot b$ and $a_t + \theta_4 \cdot b_t$ satisfy a certain integral relation; here

$$\theta_4(x) = \frac{x - \xi_4}{|x - \xi_4|}, \qquad x \in \overline{B}.$$

This relation and the proof of the uniqueness result were inspired by a relation and an argument in [5], where a similar uniqueness question was studied though in the time-independent setting.

There is a gauge invariance associated with the problem of recovering a, b, c. If $\phi(x, t)$ and $f(x, t)$ are smooth functions on $\mathbb{R}^3 \times \mathbb{R}$, we have

$$(\partial_t - a - \phi_t)\,(e^{\phi} f) = e^{\phi}\,(\partial_t - a)\,f, \quad (\nabla - b - \nabla\phi)\,(e^{\phi} f) = e^{\phi}\,(\nabla - b)\,f,$$

resulting in

$$\mathcal{L}_{a+\phi_t, b+\nabla\phi, c}(e^{\phi} f) = e^{\phi} \mathcal{L}_{a,b,c} f.$$

Hence, if $\phi(\xi, t) = 0$ for $t \in \mathbb{R}$, we have

$$\mathcal{L}_{a+\phi_t, b+\nabla\phi, c}(e^{\phi} U) = e^{\phi} \mathcal{L}_{a,b,c}(U) = 4\pi e^{\phi} H(t - \tau)\delta(x - \xi) = 4\pi H(t - \tau)\delta(x - \xi)$$

and

$$\mathcal{L}_{a+\phi_t, b+\nabla\phi, c}(e^{\phi} V) = 4\pi\delta(t - \tau, x - \xi).$$

As a consequence, $\mathcal{F}(a, b, c) = \mathcal{F}(a + \phi_t, b + \nabla\phi, c)$ for any smooth function $\phi(x, t)$ with support in $\overline{B} \times [0,T]$ and $\phi(\cdot, T) = 0, \phi_t(\cdot, T) = 0$. This suggests we can hope to recover at most the curl of $[a, b]$, that is $d(adt + b^1 dx^1 + b^2 dx^2 + b^3 dx^3)$.

Theorem 1.6 (Uniqueness for $\operatorname{curl}(a, b)$ and c)**.** *Suppose $a, c, \acute{a}, \acute{c}$ and $b, \acute{b}$ are smooth functions and vector fields on $\mathbb{R}^3 \times \mathbb{R}$ with support in $\overline{B} \times [0,T]$. Let $\xi_1, \cdots, \xi_4$ be a diverse set* *of locations with respect to* $(T+1)B$ *and* $u, \acute{u}$ *and* $v, \acute{v}$ *the functions associated with* (a,b,c) *and* $(\acute{a}, \acute{b}, \acute{c})$*, respectively, guaranteed by Propositions 1.1 and 1.2. If*

$$[u-\acute{u},(u-\acute{u})_t](x,T,\xi_i,\tau)=0, \quad \forall x \in H_{\xi_i,\tau},\ \tau \in [-1-|\xi_i|, T+1-|\xi_i|],\ i\in\{1,2,3,4\},$$
$$[v-\acute{v},(v-\acute{v})_t](x,T,\xi_4,\tau)=0, \quad \forall x \in H_{\xi_4,\tau},\ \tau \in [-1-|\xi_4|, T+1-|\xi_4|],$$

and

$$\int_0^{|x-\xi_4|} \Big((a-\acute{a})+\theta_4\cdot(b-\acute{b})\Big)(x-s\theta_4, T-s)\ \mathrm{d}s = 0, \quad \forall x \in \mathbb{R}^3,$$
$$\int_0^{|x-\xi_4|} \Big((a-\acute{a})_t+\theta_4\cdot(b-\acute{b})_t\Big)(x-s\theta_4, T-s)\ \mathrm{d}s = 0, \quad \forall x \in \mathbb{R}^3,$$

then

$$\mathrm{d}\left(a\mathrm{d}t + \sum_{i=1}^3 b^i \mathrm{d}x^i\right) = \mathrm{d}\left(\acute{a}\mathrm{d}t + \sum_{i=1}^3 \acute{b}^i \mathrm{d}x^i\right), \quad c=\acute{c}.$$

Note that we use data from the $u, \acute{u}$ solutions for all four source locations $\xi_1, \cdots, \xi_4$ but we use data from the $v, \acute{v}$ solutions only for the source at ξ_4.

We also have a Lipschitz stability result for the recovery of $\operatorname{curl}(a,b)$ and c. However, we require more data than was needed for the uniqueness result in Theorem 1.6. Let ψ be the solution of the IVP

$$\Box\psi = c - a_t + \nabla\cdot b, \quad \text{in } \mathbb{R}^3\times(-\infty,T]; \tag{1.20}$$
$$\psi(\cdot,t)=0, \quad t<0. \tag{1.21}$$

For the stability result, in addition to $\mathcal{F}(a,b,c)$, we need the traces of ψ, ψ_t, ψ_{tt} on $t=T$; this replaces the integral condition used in Theorem 1.6. We do not know whether there is stability without this extra data.

Theorem 1.7 (Stability for $\operatorname{curl}(a,b)$ and c)**.** *Suppose* $\xi_1, \cdots, \xi_4$ *is a diverse set of locations with respect to* $(T+1)B$*. Given* $M>0$*, if* $a, c, \acute{a}, \acute{c}$ *and* $b, \acute{b}$ *are smooth functions and vector fields on* $\mathbb{R}^3\times\mathbb{R}$ *with support in* $\overline{B}\times[0,T]$ *and* $\|[a,b,c,\acute{a},\acute{b},\acute{c}]\|_{C^{22}(\overline{B}\times[0,T])}\le M$*, then*

$$\begin{aligned}
&\|[\mathrm{d}\eta-\mathrm{d}\acute{\eta}, c-\acute{c}]\|_{L^2(\mathbb{R}^3\times[0,T])} \\
&\quad\preccurlyeq \sum_{i=1}^4 \int_{-1-|\xi_i|}^{T+1-|\xi_i|} \Big(\|(u-\acute{u})(\cdot,T;\xi_i,\tau)\|_{2,H_{\xi_i,\tau}} + \|(u_t-\acute{u}_t)(\cdot,T;\xi_i,\tau)\|_{1,H_{\xi_i,\tau}}\Big)\,\mathrm{d}\tau \\
&\quad+ \sum_{i=1}^4 \int_{-1-|\xi_i|}^{T+1-|\xi_i|} \|\left(u_{tt}-\acute{u}_{tt}\right)(\cdot,T;\xi_i,\tau)\|_{0,H_{\xi_i,\tau}}\ \mathrm{d}\tau \\
&\quad+ \sum_{i=1}^4 \int_{-1-|\xi_i|}^{T+1-|\xi_i|} \Big(\|(v-\acute{v})(\cdot,T;\xi_i,\tau)\|_{1,H_{\xi_i,\tau}} + \|(v_t-\acute{v}_t)(\cdot,T;\xi_i,\tau)\|_{0,H_{\xi_i,\tau}}\Big)\,\mathrm{d}\tau \\
&\quad+ \|(\psi-\acute{\psi})(\cdot,T)\|_{2,\mathbb{R}^3} + \|(\psi_t-\acute{\psi}_t)(\cdot,T)\|_{1,\mathbb{R}^3} + \|(\psi_{tt}-\acute{\psi}_{tt})(\cdot,T)\|_{0,\mathbb{R}^3},
\end{aligned}$$

where η and $\acute{\eta}$ are the 1-forms

$$\eta = a\mathrm{d}t + \sum_{i=1}^{3} b^i \mathrm{d}x^i, \quad \acute{\eta} = \acute{a}\mathrm{d}t + \sum_{i=1}^{3} \acute{b}^i \mathrm{d}x^i$$

and the constant is independent of $a, b, c, \acute{a}, \acute{b}, \acute{c}$. Here ψ, ψ' are the solutions of the IVP (1.20), (1.21) and $u, \acute{u}, v, \acute{v}$ are the functions corresponding to (a, b, c) and $(\acute{a}, \acute{b}, \acute{c})$ guaranteed by Propositions 1.1 and 1.2 .

A fundamental aspect of our work is the Lipschitz stability results for the space and time dependent coefficients obtained by the use of Carleman estimates on domains depending on the parameter τ and the integral of these estimates with respect to τ. We very much exploit the relation between the unknown coefficients and the traces of the solutions of the IVP on the characteristic cones.

We introduce some notation used in the rest of the article. For convenience, we denote the operators $\mathcal{L}_{a,b,c}$ and $\mathcal{L}_{\acute{a},\acute{b},\acute{c}}$ by $\mathcal{L}$ and $\acute{\mathcal{L}}$ respectively. We define the differences

$$\overline{a} := a - \acute{a}, \quad \overline{b} := b - \acute{b}, \quad \overline{c} := c - \acute{c}, \quad \overline{q} := q - \acute{q}, \quad \overline{u} := u - \acute{u}, \quad \overline{v} := v - \acute{v}. \tag{1.22}$$

Also, given a $\xi \in \mathbb{R}^3 \setminus \overline{B}$ and $x \in \overline{B}$, recall that we have defined

$$\theta(x) := \frac{x - \xi}{|x - \xi|}, \qquad x \in \mathbb{R}^3,\ x \neq \xi.$$

We use θ instead of $\theta(x)$ most of the time and we use θ_i when ξ is replaced by ξ_i.

A key ingredient of the proofs of the theorems is a Carleman estimate, with explicit boundary terms, for the operator $\mathcal{L}_{a,b,c}$, in the region $Q_{\xi,\tau}$. We state it here and give the proof in Section 7.

Proposition 1.8. *Suppose $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}^3$ and a, q, b are smooth functions and vector fields in $\mathbb{R}^3 \times \mathbb{R}$ with $\{\xi\} \times \mathbb{R}$ disjoint from the supports of a, b, q. Then there is a $\sigma_0 > 0$ so that*

$$\begin{aligned}\sigma \int_{Q_{\xi,\tau}} e^{2\sigma t} \left(|\nabla_{x,t} w|^2 + \sigma^2 w^2\right) + \sigma \int_{C_{\xi,\tau}} e^{2\sigma t} \left(|\nabla_C w|^2 + \sigma^2 w^2\right) \\ \preccurlyeq \left(\int_{Q_{\xi,\tau}} e^{2\sigma t} |\mathcal{L}_{a,b,q} w|^2 + \sigma \int_{H_{\xi,\tau}} e^{2\sigma t} \left(|\nabla_{x,t} w|^2 + \sigma^2 w^2\right) \right),\end{aligned} \tag{1.23}$$

for every $w \in C^3(Q_{\xi,\tau})$ and every $\sigma \geq \sigma_0$. Here ∇_C represents the gradient on the submanifold $C_{\xi,\tau}$. Further, the constant is independent of w and σ and depends only on $T, |\xi|, |\tau|$ and $\|[a, b, q]\|_{C^0(Q_{\xi,\tau})}$.

The rest of the article gives the proofs of the propositions and the theorems stated above.

2. Proof of Theorem 1.3

We have $a = \acute{a},\ b = \acute{b}$ and a single source $\xi \in \mathbb{R}^3 \setminus \overline{B}$. For any $\tau \in [-(1+|\xi|), T+1-|\xi|]$, let $v, \acute{v}$ be the functions guaranteed by Proposition 1.2 for the coefficients (a, b, q) and $(a, b, \acute{q})$. Taking the differences of (1.17), (1.18) for the two sets of coefficients, we obtain

$$\mathcal{L}\overline{v} = -\overline{q}\acute{v}, \quad \text{in } Q_{\xi,\tau}, \tag{2.1}$$

$$2\left(\partial_t + \theta \cdot \nabla - (a + \theta \cdot b) + r^{-1}\right)\overline{v} = -\overline{q}\alpha, \quad \text{on } C_{\xi,\tau}. \tag{2.2}$$

Applying Proposition 1.8 to the function $\overline{v}$ in the region $Q_{\xi,\tau}$, we have

$$\sigma\|\overline{v}\|^2_{1,\sigma,C_{\xi,\tau}} \preccurlyeq \|\mathcal{L}\overline{v}\|^2_{0,\sigma,Q_{\xi,\tau}} + \sigma\|\overline{v}\|^2_{1,\sigma,H_{\xi,\tau}} + \sigma\|\overline{v}_t\|^2_{0,\sigma,H_{\xi,\tau}},$$

with the constant dependent only on $|\xi|, T,\ \|[a,b,q]\|_{C^0(Q_{\xi,\tau})}$. Hence, using (2.1), (2.2) and that $r\alpha$ is a positive continuous function on $\overline{Q}_{\xi,\tau}$, we have

$$\begin{aligned}\sigma\|\overline{q}\|^2_{0,\sigma,C_{\xi,\tau}} &\preccurlyeq \sigma\|r\alpha\overline{q}\|^2_{0,\sigma,C_{\xi,\tau}} \preccurlyeq \sigma\|r(\partial_t + \theta\cdot\nabla - (a+\theta\cdot b) + r^{-1})\overline{v}\|^2_{0,\sigma,C_{\xi,\tau}} \\ &\preccurlyeq \sigma\|\overline{v}\|^2_{1,\sigma,C_{\xi,\tau}} \preccurlyeq \|\overline{q}\|^2_{0,\sigma,Q_{\xi,\tau}} + \sigma\|\overline{v}\|^2_{1,\sigma,H_{\xi,\tau}} + \sigma\|\overline{v}_t\|^2_{0,\sigma,H_{\xi,\tau}}.\end{aligned} \tag{2.3}$$

Here the constant depends on $|\xi|, T,\ \|[a,b,q]\|_{C^0(Q_{\xi,\tau})}$ and $\|\acute{v}\|_{C^0}$. Using Proposition 1.2, the constant depends only on T, M and $|\xi|$.

Noting that $q, \acute{q}$ are supported in $\overline{B}\times[0,T]$ and $\overline{B}\times[0,T]$ does not intersect $C_{\xi,\tau}$ if τ is outside the interval $[-(1+|\xi|), T+1-|\xi|]$, we have

$$\begin{aligned}\int_{-1-|\xi|}^{T+1-|\xi|}\int_{C_{\xi,\tau}} e^{2\sigma t}\,|\overline{q}(x,t)|^2\ \mathrm{d}S\ \mathrm{d}\tau &= \int_{\mathbb{R}}\int_{C_{\xi,\tau}} e^{2\sigma t}\,|\overline{q}(x,t)|^2\ \mathrm{d}S\ \mathrm{d}\tau \\ &= \sqrt{2}\int_{\mathbb{R}}\int_{\mathbb{R}^3\times\mathbb{R}} e^{2\sigma t}\,|\overline{q}(x,t)|^2\,\delta(t-\tau-|x-\xi|)\ \mathrm{d}x\mathrm{d}t\ \mathrm{d}\tau \\ &= \sqrt{2}\int_{\mathbb{R}^3\times\mathbb{R}} e^{2\sigma t}\,|\overline{q}(x,t)|^2\int_{\mathbb{R}}\delta(t-\tau-|x-\xi|)\ \mathrm{d}\tau\ \mathrm{d}x\mathrm{d}t \\ &= \sqrt{2}\int_{\mathbb{R}^3\times\mathbb{R}} e^{2\sigma t}\,|\overline{q}(x,t)|^2\ \mathrm{d}x\mathrm{d}t \\ &= \sqrt{2}\,\|\overline{q}\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]}.\end{aligned}$$

Hence integrating (2.3) w.r.t τ over the interval $[-1-|\xi|, T+1-|\xi|]$, we obtain

$$\sigma\|\overline{q}\|^2_{0,\sigma,\overline{B}\times[0,T]} \preccurlyeq \|\overline{q}\|^2_{0,\sigma,\overline{B}\times[0,T]} + \sigma\int_{-1-|\xi|}^{T+1-|\xi|}\left(\|\overline{v}\|^2_{1,\sigma,H_{\xi,\tau}} + \|\overline{v}_t\|^2_{0,\sigma,H_{\xi,\tau}}\right)\ \mathrm{d}\tau,$$

which proves Theorem 1.3 if we take σ to be large enough.

3. Proof of Theorem 1.5

Fix a ξ_i and a $\tau \in [-(1+|\xi_i|), T+1-|\xi_i|]$. Let $u, \acute{u}$ be the solutions corresponding to the coefficients (a,b,q) and $(\acute{a},\acute{b},q)$ guaranteed by Proposition 1.1. Note that $u, \acute{u}$ are zero in a neighborhood of $(x{=}\xi_i, t{=}\tau)$ and $a, b, q, \acute{a}, \acute{b}, \acute{q}$ are supported away from $\{\xi\} \times \mathbb{R}$.

Taking the differences of versions of (1.13), (1.14) associated to (a,b,q) and $(\acute{a},\acute{b},q)$, for each ξ_i, we have

$$\mathcal{L}\overline{u} = 2\overline{a}\acute{u}_t - 2\overline{b}\cdot\nabla\acute{u} + \frac{2\overline{b}\cdot(x-\xi_i)}{|x-\xi_i|^3}, \qquad \text{in } Q_{\xi_i,\tau}, \tag{3.1}$$

$$\overline{u} = \alpha - \acute{\alpha}, \qquad \text{on } C_{\xi_i,\tau}. \tag{3.2}$$

Applying Proposition 1.8 to $\overline{u}$ in the region $Q_{\xi_i,\tau}$, we have

$$\sigma\|\overline{u}\|^2_{1,\sigma,C_{\xi_i,\tau}} \preccurlyeq \|\mathcal{L}\overline{u}\|^2_{0,\sigma,Q_{\xi_i,\tau}} + \sigma\|\overline{u}\|^2_{1,\sigma,H_{\xi_i,\tau}} + \sigma\|\overline{u}_t\|^2_{0,\sigma,H_{\xi_i,\tau}}, \tag{3.3}$$

with the constant dependent only on $\xi_i, T, \|[a,b,q]\|_{C^0}$.

Now, using (1.10), we have

$$\begin{aligned}(\partial_t + \theta\cdot\nabla - (a+\theta_i\cdot b) + r^{-1})(\alpha-\acute{\alpha}) & \\ &= -(\partial_t + \theta\cdot\nabla - (a+\theta_i\cdot b) + r^{-1})\acute{\alpha} \\ &= -(\partial_t + \theta\cdot\nabla - (\acute{a}+\theta_i\cdot\acute{b}) + r^{-1})\acute{\alpha} - (\overline{a}+\theta_i\cdot\overline{b})\acute{\alpha} \\ &= -(\overline{a}+\theta_i\cdot\overline{b})\acute{\alpha}.\end{aligned} \tag{3.4}$$

So, using (3.1) - (3.4) and noting that $\overline{b}$ is supported away from $\{\xi_i\}\times\mathbb{R}$, we obtain

$$\sigma\|\overline{a}+\theta_i\cdot\overline{b}\|^2_{0,\sigma,C_{\xi_i,\tau}} \preccurlyeq \sigma\|\overline{u}\|^2_{1,\sigma,C_{\xi_i,\tau}} \preccurlyeq \|[\overline{a},\overline{b}]\|^2_{0,\sigma,Q_{\xi_i,\tau}} + \sigma\|\overline{u}\|^2_{1,\sigma,H_{\xi_i,\tau}} + \sigma\|\overline{u}_t\|^2_{0,\sigma,H_{\xi_i,\tau}}, \tag{3.5}$$

with the constant dependent only on $\xi_i, T, \|[a,b,q]\|_{C^0}$ and $\|\acute{u}\|_{C^1}$, hence on $|\xi_4|, T$ and M.

Imitating the integral relation calculation in the proof of Theorem 1.3, we have

$$\int_{-1-|\xi_i|}^{T+1-|\xi_i|}\int_{C_{\xi_i,\tau}} e^{2\sigma t}\,|\overline{a}+\theta_i\cdot\overline{b}|^2\,dS_{x,t}\,d\tau = \sqrt{2}\int_{\overline{B}\times[0,T]} e^{2\sigma t}\,|\overline{a}+\theta_i\cdot\overline{b}|^2\,dx\,dt.$$

Hence integrating (3.5) w.r.t τ over $[-1-|\xi_i|, T+1-|\xi_i|]$

$$\sigma\|\overline{a}+\theta_i\cdot\overline{b}\|^2_{0,\sigma,\overline{B}\times[0,T]} \preccurlyeq \|[\overline{a},\overline{b}]\|^2_{0,\sigma,\overline{B}\times[0,T]} + \sigma\int_{-1-|\xi_i|}^{T+1-|\xi_i|}\left(\|\overline{u}\|^2_{1,\sigma,H_{\xi_i,\tau}} + \|\overline{u}_t\|^2_{0,\sigma,H_{\xi_i,\tau}}\right)\,\mathrm{d}\tau. \tag{3.6}$$

Using (3.6) for each $i=1,2,3,4$ and noting that ξ_i, $i=1,\cdots,4$, is a diverse set of locations w.r.t B, we obtain

$$\sigma\|[\overline{a},\overline{b}]\|^2_{0,\sigma,\overline{B}\times[0,T]} \preccurlyeq \|[\overline{a},\overline{b}]\|^2_{0,\sigma,\overline{B}\times[0,T]} + \sigma\sum_{i=1}^{4}\int_{-1-|\xi_i|}^{T+1-|\xi_i|}\left(\|\overline{u}\|^2_{1,\sigma,H_{\xi_i,\tau}} + \|\overline{u}_t\|^2_{0,\sigma,H_{\xi_i,\tau}}\right)\,\mathrm{d}\tau$$

The theorem follows if we choose σ large enough.

4. Proof of Theorem 1.6

Our proof borrows an idea from [5]. We seek to prove the uniqueness in the recovery $\mathrm{curl}(a,b)$ and c from the values of U, U_t, V, V_t on $H_{\xi,\tau}$ for $\xi = \xi_i$, $i = 1, \cdots, 4$.

The $u, \acute{u}, v, \acute{v}$ are the solutions guaranteed by Propositions 1.1 and 1.2 for the coefficients a, b, c and $\acute{a}, \acute{b}, \acute{c}$. Since a, b, c are supported in $\overline{B} \times [0,T]$ and $\xi \in \mathbb{R}^3 \setminus (T+1)\overline{B}$, one may check that for a fixed $\xi \in (T+1)B$, the values of u, u_t, v, v_t and $\acute{u}, \acute{u}_t, \acute{v}, \acute{v}_t$ on $\mathbb{R}^3 \times \{t{=}T\}$ are zero for $\tau > T+1-|\xi|$ and do not change as τ varies over $\tau \in (-\infty, -1-|\xi|]$. Hence, from the hypothesis of Theorem 1.6, we may assume that

$$[u-\acute{u}, (u-\acute{u})_t](x, T, \xi_i, \tau) = 0, \quad \forall x \in H_{\xi_i,\tau},\ i \in \{1,2,3,4\},$$
$$[v-\acute{v}, (v-\acute{v})_t](x, T, \xi_4, \tau) = 0, \quad \forall x \in H_{\xi_4,\tau},$$

for all $\tau \in \mathbb{R}$ rather than for a limited range of τ.

As discussed in the introduction, due to gauge invariance, there is a natural obstruction to uniqueness when attempting to recover a, b, c. For $x \neq \xi_4$, define

$$\phi(x,t) := -\int_0^{|x-\xi_4|} (a + \theta_4 \cdot b)(x - s\theta_4, t-s)\, \mathrm{d}s,$$
$$\acute{\phi}(x,t) = -\int_0^{|x-\xi_4|} (\acute{a} + \theta_4 \cdot \acute{b})(x - s\theta_4, t-s)\, \mathrm{d}s.$$

Since $\xi_4 \notin (T+1)\overline{B}$ and $a, b, c, \acute{a}, \acute{b}, \acute{c}$ are supported in $\overline{B} \times [0,T]$, we have $\phi = 0$ and $\acute{\phi} = 0$ in a punctured cylindrical neighborhood of $\{\xi_4\} \times \mathbb{R}$. Hence, defining $\phi(\xi_4, \cdot) := 0$, $\acute{\phi}(\xi_4, \cdot) := 0$ gives us smooth functions on $\mathbb{R}^3 \times \mathbb{R}$ which are zero in a cylindrical neighborhood of $\{0\} \times \mathbb{R}$. We also note that the intersection of the supports of $\phi, \acute{\phi}$ with $\mathbb{R}^3 \times (-\infty, T]$ is contained in $(T+1)\overline{B} \times (-\infty, T]$.

As shown in the introduction, $e^{\phi}u$, $e^{\acute{\phi}}\acute{u}$ $e^{\phi}v$, $e^{\acute{\phi}}\acute{v}$ are the functions guaranteed by Propositions 1.1 and 1.2 for the coefficients $a + \phi_t, b + \nabla\phi, c$ and $\acute{a} + \acute{\phi}_t, \acute{b} + \nabla\acute{\phi}, \acute{c}$, provided the hypotheses of these propositions are satisfied. The propositions require that the cylinder $\{\xi\} \times \mathbb{R}$ not intersect the supports of the coefficients, which seems not to be true for the modified a, b, c. However, we will be using the values of $e^{\phi}u$, $e^{\acute{\phi}}\acute{u}$ $e^{\phi}v$, $e^{\acute{\phi}}\acute{v}$ only on subsets of the region $\mathbb{R}^3 \times (-\infty, T]$, so we need Propositions 1.1 and 1.2 only for the region $\mathbb{R}^3 \times (-\infty, T]$. Since $|\xi_i| > T+1$, the cylinders $\{\xi_i\} \times (-\infty, T]$, $i = 1,2,3,4$, do not intersect the supports of $a + \phi_t, b + \nabla\phi, c$ and $\acute{a} + \acute{\phi}_t, \acute{b} + \nabla\acute{\phi}, \acute{c}$, so Propositions 1.1 and 1.2 are valid on $\mathbb{R}^3 \times (-\infty, T]$ for the coefficients $a + \phi_t, b + \nabla\phi, c$ and $\acute{a} + \acute{\phi}_t, \acute{b} + \nabla\acute{\phi}, \acute{c}$.

From our hypothesis, we have

$$\phi(x,T) = \acute{\phi}(x,T),\ \phi_t(x,T) = \acute{\phi}_t(x,T)$$

Hence, on $H_{\xi_i,\tau}$, $i = 1, 2, 3, 4$, we have

$$[e^{\phi}u, (e^{\phi}u)_t](\cdot, T, \xi_i, \tau)] = [e^{\acute{\phi}}\acute{u}, (e^{\acute{\phi}}\acute{u})_t](\cdot, T, \xi_i, \tau)], \quad \forall \tau \in \mathbb{R},$$
$$[e^{\phi}v, (e^{\phi}v)_t](\cdot, T, \xi_4, \tau)] = [e^{\acute{\phi}}\acute{v}, (e^{\acute{\phi}}\acute{v})_t](\cdot, T, \xi_4, \tau)], \quad \forall \tau \in \mathbb{R}.$$

Thus to prove Theorem 1.6, it suffices to work with the modified coefficients $(a+\phi_t, b+\nabla\phi, c)$ and $(\acute{a} + \acute{\phi}_t, \acute{b} + \nabla\acute{\phi}, \acute{c})$ because $(a + \phi_t, b + \nabla\phi)$ has same curl as (a, b) and $(\acute{a} + \acute{\phi}_t, \acute{b} + \nabla\acute{\phi})$ has the same curl as $(\acute{a}, \acute{b})$. Further, since $a, b, \acute{a}, \acute{b}$ are supported away from $\{\xi\} \times \mathbb{R}$, we have

$$\begin{aligned}(\partial_t + \theta_4 \cdot \nabla)\, \phi &= \int_0^{|x-\xi_4|} \frac{\mathrm{d}}{\mathrm{d}s}(a + \theta_4 \cdot b)(x - s\theta_4, t - s)\; \mathrm{d}s \\ &= -(a + \theta_4 \cdot b)(x, t)\end{aligned}$$

which implies

$$(a + \phi_t) + \theta_4 \cdot (b + \nabla\phi) = 0, \qquad (\acute{a} + \acute{\phi}_t) + \theta_4 \cdot (\acute{b} + \nabla\acute{\phi}) = 0$$

So, to prove Theorem 1.6, we may assume that

$$a + \theta_4 \cdot b = \acute{a} + \theta_4 \cdot \acute{b} = 0. \tag{4.1}$$

We show that $(a, b, q) = (\acute{a}, \acute{b}, \acute{q})$ for these modified triples, which will prove the theorem. There is a subtle point here which we discuss next.

We replace a, b (and do the same for $\acute{a}, \acute{b}$) by a special a, b with $a + \theta_4 \cdot b = 0$ but with the same curl as the original a, b and with the corresponding (modified) values of u, u_t, v, v_t and $\acute{u}, \acute{u}_t, \acute{v}, \acute{v}_t$ agreeing on $H_{\xi,\tau}$ as for the original. However we do not modify $c, \acute{c}$. If we prove uniqueness in the recovery of this modified a, b and $q = c - a_t + \nabla \cdot b + |a|^2 - |b|^2$ then we do not necessarily have uniqueness in the recovery of the original curl(a, b) and q. However, uniqueness in the recovery of the modified a, b, q also gives us uniqueness in the recovery of the c and we have not changed c. So we have uniqueness in the recovery of the original curl(a, b) and c.

Define

$$\tau_{min} := \min\{-(|\xi_i| + T + 1) : i = 1, 2, 3, 4\}, \qquad \tau_{max} := \max\{2T + 1 - |\xi_i| : i = 1, 2, 3, 4\},$$

and our τ will vary in the interval $[\tau_{min}, \tau_{max}]$. For use below, observe that the intersection of the supports of the modified a, b, q with $\mathbb{R}^3 \times (-\infty, T]$ is contained in $(T + 1)\overline{B} \times [0, T]$ and the union of the $C_{\xi,\tau}$, as τ varies over $[\tau_{min}, \tau_{max}]$, contains $(T + 1)\overline{B} \times [0, T]$.

From (1.13) - (1.14), for each $i = 1, \cdots, 4$, we obtain

$$\begin{aligned}\mathcal{L}\overline{u} = 2\overline{a}\acute{u}_t - 2\overline{b} \cdot \nabla\acute{u} - \overline{q}\acute{u} + \frac{2\overline{b} \cdot (x - \xi_i)}{|x - \xi_i|^3} - \frac{\overline{q}(x, t)}{|x - \xi_i|}, &\quad \text{in } Q_{\xi_i,\tau}, \\ \overline{u} = \alpha - \acute{\alpha}, &\quad \text{on } C_{\xi_i,\tau}.\end{aligned} \tag{4.2}$$

Noting that $\{\xi_i\} \times (-\infty, T]$ does not intersect the supports of $\overline{b}$ and $\overline{q}$, we have

$$\|\mathcal{L}\overline{u}\|_{0,\sigma,Q_{\xi,\tau}} \preccurlyeq \|[\overline{a}, \overline{b}, \overline{q}]\|_{0,\sigma,Q_{\xi_i,\tau}}.$$

So applying Proposition 1.8 to $\overline{u}$ on the region $Q_{\xi_i,\tau}$, using 4.2, and proceeding as in the proof of Theorem 1.5 including the fact that the locations $\xi_1, \cdots, \xi_4$ are diverse w.r.t $(T+1)B$, for large enough σ we obtain

$$\sigma \|[\overline{a}, \overline{b}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \ \leq \ C_1 \left(\|[\overline{a}, \overline{b}, \overline{q}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \right). \tag{4.3}$$

Note the $\overline{q}$ term on the RHS of (4.3). This was absent from the RHS of the similar inequality, when proving Theorem 1.5, because $\overline{q} = 0$ for Theorem 1.5.

Next we estimate the norm of $\overline{q}$ using the data coming from $v, \acute{v}$. Taking the differences of (1.17), (1.18) for the coefficients (a, b, q) and $(\acute{a}, \acute{b}, \acute{q})$, we have

$$\mathcal{L}\overline{v}_4 = 2\overline{a}\,\acute{v}_{4t} - 2\overline{b}\cdot\nabla\acute{v}_4 - \overline{q}\acute{v}_4, \quad \text{in } Q_{\xi_4,\tau}, \tag{4.4}$$

$$2(\partial_t + \theta_4\cdot\nabla - (a+\theta_4\cdot b) + r^{-1})\overline{v} = \acute{\mathcal{L}}\acute{\alpha}_4 - \mathcal{L}\alpha_4 + 2(\overline{a}+\theta_4\cdot\overline{b})\acute{v}_4, \quad \text{on } C_{\xi_4,\tau}. \tag{4.5}$$

Since $a + \theta_4\cdot b = \acute{a} + \theta_4\cdot\acute{b} = 0$, we have

$$\alpha_4(x,t) = \acute{\alpha}_4(x,t) = \frac{1}{|x-\xi_4|},$$

hence, for $x \neq \xi_4$,

$$\acute{\mathcal{L}}\acute{\alpha}_4 - \mathcal{L}\alpha_4 = -\left(\mathcal{L} - \acute{\mathcal{L}}\right)\alpha_4 = \frac{\overline{b}(x,t)\cdot(x-\xi_4)}{|x-\xi_4|^3} - \frac{\overline{q}(x,t)}{|x-\xi_4|},$$

so (4.5) gives us

$$2\left(\partial_t + \theta_4\cdot\nabla - (a+\theta_4\cdot b) + r^{-1}\right)\overline{v} = \frac{2\overline{\theta}_4\cdot\overline{b}}{|x-\xi_4|^2} - \frac{\overline{q}(x,t)}{|x-\xi_4|}, \qquad \text{on } C_{\xi_4,\tau}. \tag{4.6}$$

From (4.6), noting that $\{\xi_4\}\times(-\infty,T]$ does not intersect the supports of $\overline{a}, \overline{b}, \overline{q}$ and $\overline{v}$, we have

$$|\overline{q}| \preccurlyeq |\overline{b}| + |\nabla_C\,\overline{v}| + |\overline{v}|, \qquad \text{on } C_{\xi_4,\tau}$$

where ∇_C is the gradient on $C_{\xi_4,\tau}$ and the constant is dependent only on T, ξ_4 and $\|[a, b, \acute{v}_4]\|_{C^0}$, hence only on T, M and $|\xi_4|$. Therefore

$$\|\overline{q}\|_{0,\sigma,C_{\xi_4,\tau}} \preccurlyeq \|[\overline{a}, \overline{b}]\|_{0,\sigma,C_{\xi_4,\tau}} + \|\overline{v}\|_{1,\sigma,C_{\xi_4,\tau}}.$$

Also, from (4.4), we have

$$\|\mathcal{L}\overline{v}_4\|_{0,\sigma,Q_{\xi_4,\tau}} \preccurlyeq \|[\overline{a}, \overline{b}, \overline{q}]\|_{0,\sigma,Q_{\xi_4,\tau}}$$

with the constant dependent only on T and $\|[\acute{v}_4]\|_{C^1}$, hence dependent only on T and M. Using these observations in Proposition 1.8 applied to $\overline{v}$ on the region $Q_{\xi_4,\tau}$ and noting that $\overline{v}, \overline{v}_t$ are zero on $H_{\xi_4,\tau}$, we obtain

$$\sigma\|\overline{q}\|^2_{0,\sigma,C_{\xi_4,\tau}} \preccurlyeq \|[\overline{a}, \overline{b}, \overline{q}\|^2_{0,\sigma,Q_{\xi_4,\tau}} + \sigma\|[\overline{a}, \overline{b}]\|^2_{0,\sigma,C_{\xi_4,\tau}}, \qquad \forall \tau \in [\tau_{min}, \tau_{max}],$$

for σ large enough. Integrating this inequality w.r.t τ, over the interval $[\tau_{min}, \tau_{max}]$, using integral relations similar to the one in the proof of Theorem 1.3, we obtain

$$\sigma \|\overline{q}\|^2_{\mathbb{R}^3\times[0,T]} \leq C_2 \left(\|\overline{q}\|^2_{\mathbb{R}^3\times[0,T]} + \sigma \|[\overline{a}, \overline{b}]\|^2_{\mathbb{R}^3\times[0,T]} \right),$$

for σ large enough. So adding to this a $C_2 + 1$ multiple of (4.3) we obtain

$$\sigma \|[\overline{a}, \overline{b}, \overline{q}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \preccurlyeq \|[\overline{a}, \overline{b}, \overline{q}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]},$$

for σ large enough. Hence, choosing σ large enough, we obtain $\overline{a} = 0, \overline{b} = 0, \overline{q} = 0$, proving the theorem.

5. Proof of Theorem 1.7

The $u, \acute{u}, v, \acute{v}$ are the solutions guaranteed by Propositions 1.1 and 1.2 for the coefficients a, b, c and $\acute{a}, \acute{b}, \acute{c}$. Since a, b, c are supported in $\overline{B} \times [0, T]$ and $\xi \in \mathbb{R}^3 \setminus (T+1)\overline{B}$, one may check that for a fixed $\xi \in (T+1)B$, the values of u, u_t, u_{tt}, v, v_t and $\acute{u}, \acute{u}_t, \acute{u}_{tt}, \acute{v}, \acute{v}_t$ on $\mathbb{R}^3 \times \{t{=}T\}$ are zero for $\tau > T + 1 - |\xi|$ and do not change as τ varies over $\tau \in (-\infty, -1 - |\xi|]$. Hence, in the statement of Theorem 1.7, the τ integrals may be replaced by τ integrals over any finite interval that contains $[-1 - |\xi_i|, T + 1 - |\xi_i|]$.

We note that $\psi(x, t)$ is a smooth function on $\mathbb{R}^3 \times \mathbb{R}$ and its support intersects $\mathbb{R}^3 \times (-\infty, T]$ in a region contained in $(T + 1)\overline{B} \times (-\infty, T]$.

We recall from the first section that, if U and V are the solutions, corresponding to the a, b, c, guaranteed by Propositions 1.1 and 1.2, then $e^{\psi}U$ and $e^{\psi}V$ are the solutions, corresponding to the coefficients $a + \psi_t, b + \nabla\psi, c$, guaranteed by Propositions 1.1 and 1.2. Further one may verify that $[a, b]$ and $[a + \psi_t, b + \nabla\psi]$ have the same curl and

$$c - (a + \psi_t)_t + \nabla \cdot (b + \nabla\psi) = 0.$$

We also observe that the intersection of the supports of $a + \psi_t, b + \nabla\psi, c$, with $\mathbb{R}^3 \times (-\infty, T]$ are contained in $(T + 1)\overline{B} \times (-\infty, T]$. In particular, $\{\xi_i\} \times (-\infty, T]$ does not intersect the supports of $a + \psi_t, b + \nabla\psi, c$ so Propositions 1.1 and 1.2 apply even to these modified a, b, c.

For bounded functions $\psi, \acute{\psi}, w, \acute{w}$, we have

$$\begin{aligned} \left| e^{\psi} w - e^{\acute{\psi}} \acute{w} \right| &\leq \left| e^{\psi}(w - \acute{w}) \right| + \left| \left(e^{\psi} - e^{\acute{\psi}} \right) \acute{w} \right| \\ &\leq C \left(|w - \acute{w}| + \left| e^{\psi} - e^{\acute{\psi}} \right| \right) \\ &\leq C \left(|w - \acute{w}| + |\psi - \acute{\psi}| \right), \end{aligned}$$

and one has similar estimates for the first and second order derivatives of $e^{\psi}w, e^{\acute{\psi}}\acute{w}$ also.

Keeping the above observations in mind, it is enough to prove Theorem 1.7 for a, b, c and $\acute{a}, \acute{b}, \acute{c}$ which are supported in $(T+1)\overline{B} \times (-\infty, T]$ for which

$$c - a_t + \nabla \cdot b = 0, \quad \acute{c} - \acute{a}_t + \nabla \cdot \acute{b} = 0,$$

and the τ integrals in the statement of Theorem 1.7 are over $[\tau_{min}, \tau_{max}]$ where

$$\tau_{min} := \min\{-(|\xi_i| + T + 1) : i = 1,2,3,4\}, \qquad \tau_{max} := \max\{2T + 1 - |\xi_i| : i = 1,2,3,4\}.$$

For use below, observe that the union of the $C_{\xi,\tau}$, as τ varies over $[\tau_{min}, \tau_{max}]$, contains $(T+1)\overline{B} \times [0, T]$, so, in particular, it contains the intersection of $\mathbb{R}^3 \times (-\infty, T]$ with the supports of the modified $a, b, c, \acute{a}, \acute{b}, \acute{c}$.

Note that $\psi = 0$ and $\acute{\psi} = 0$ for these modified a, b, c and $\acute{a}, \acute{b}, \acute{c}$ and the operators $\mathcal{L}$ and $\acute{\mathcal{L}}$ become

$$\mathcal{L} = \Box - 2a\partial_t + 2b \cdot \nabla + a^2 - b^2, \quad \acute{\mathcal{L}} = \Box - 2\acute{a}\partial_t + 2\acute{b} \cdot \nabla + \acute{a}^2 - \acute{b}^2.$$

To keep the expressions simple, while we work with a fixed but arbitrary $\xi \in \mathbb{R}^3 \setminus (T+1)\overline{B}$ and $\tau \in [\tau_{min}, \tau_{max}]$, we suspend showing the dependence on ξ, τ. We write $Q_{\xi,\tau}$, $H_{\xi,\tau}$ and $C_{\xi,\tau}$ as Q, H and C. Further, we define

$$D = \nabla_{x,t}, \qquad r(x) = |x - \xi|, \qquad \theta(x) = \frac{x - \xi}{|x - \xi|}, \qquad x \neq \xi.$$

and we write them as r, θ.

We have

$$(a^2 - |b|^2) - (\acute{a}^2 - |\acute{b}|^2) = (a + \acute{a})(a - \acute{a}) - (b + \acute{b}) \cdot (b - \acute{b}) = (a + \acute{a})\overline{a} - (b + \acute{b}) \cdot \overline{b}.$$

Hence $\overline{u} := u - \acute{u}$ is a solution of the characteristic BVP

$$\mathcal{L}\overline{u} = 2\overline{a}\acute{u}_t - 2\overline{b} \cdot \nabla \acute{u} + 2r^{-2}\theta \cdot \overline{b} + \left((b + \acute{b})\overline{b} - (a + \acute{a})\overline{a}\right)\left(\acute{u} + r^{-1}\right), \quad \text{in } Q, \tag{5.1}$$

$$\overline{u} = \alpha - \acute{\alpha}, \quad \text{on } C. \tag{5.2}$$

So applying Proposition 1.8 to $\overline{u}$ on Q, we obtain

$$\sigma\left(\|\overline{u}\|^2_{1,\sigma,Q} + \|\alpha - \acute{\alpha}\|^2_{1,\sigma,C}\right) \preccurlyeq \|[\overline{a}, \overline{b}]\|^2_{0,\sigma,Q} + \|\overline{u}\|^2_{1,\sigma,H} + \|\partial_t \overline{u}\|^2_{0,\sigma,H}. \tag{5.3}$$

Next, we obtain higher order estimates on $\overline{u}$ by differentiating (5.1) in directions tangential to C. If we write $x = (x^1, x^2, x^3)$ and $\xi = (\xi^1, \xi^2, \xi^3)$, the vector fields

$$\partial_t + \theta \cdot \nabla, \qquad \Omega_{lm} = (x^l - \xi^l)\partial_m - (x^m - \xi^m)\partial_l, \; l, m = 1, 2, 3,$$

span the tangent space to C at any point on C. Differentiating (5.1), (5.2) by Ω_{lm} we obtain

$$\begin{aligned}\mathcal{L}\left(\Omega_{lm}\overline{u}\right) = \Omega_{lm}&\left(2\overline{a}\acute{u}_t - 2\overline{b} \cdot \nabla\acute{v} + 2r^{-2}\theta \cdot \overline{b} + \left((b + \acute{b})\overline{b} - (a + \acute{a})\overline{a}\right)\left(\acute{u} + r^{-1}\right)\right) \\ &+ [\mathcal{L}, \Omega_{lm}]\overline{u}, \quad \text{in } Q, \end{aligned} \tag{5.4}$$

$$\Omega_{lm}(\overline{u}) = \Omega_{lm}\left(\alpha - \acute{\alpha}\right), \quad \text{on } C. \tag{5.5}$$

Since principal part of $\mathcal{L}$ is a constant coefficient operator, the operator $[\mathcal{L}, \Omega_{lm}]$ is a first order operator, hence (5.4) implies

$$|\mathcal{L}\left(\Omega_{lm}\overline{u}\right)| \preccurlyeq \left|[\overline{a}, \overline{b}, D\overline{a}, D\overline{b}]\right| + |[\overline{u}, D\overline{u}]|\,, \qquad \text{on } Q. \tag{5.6}$$

Hence, using (5.5) and (5.6) and Proposition 1.8 applied to $\Omega_{lm}\overline{u}$ on Q, we obtain

$$\begin{aligned}\sigma\|\Omega_{lm}(\alpha-\acute{\alpha})\|^2_{1,\sigma,C} \preccurlyeq \|[\overline{a}, \overline{b}, D\overline{a}, D\overline{b}]\|^2_{0,\sigma,Q} + \|[\overline{u}, D\overline{u}]\|^2_{0,\sigma,Q}\\ + \sigma\|\Omega_{lm}\overline{u}\|^2_{1,\sigma,H} + \sigma\|\partial_t\left(\Omega_{lm}\overline{u}\right)\|^2_{0,\sigma,H}.\end{aligned} \tag{5.7}$$

Using a similar argument for the vector field $\partial_t + \theta\cdot\nabla$, we obtain

$$\begin{aligned}\sigma\|(\partial_t + \theta\cdot\nabla)(\alpha-\acute{\alpha})\|^2_{1,\sigma,C} \preccurlyeq \|[\overline{a}, \overline{b}, D\overline{a}, D\overline{b}]\|^2_{0,\sigma,Q} + \|[\overline{u}, D\overline{u}]\|^2_{0,\sigma,Q}\\ + \sigma\|[\overline{u}_t, \nabla\overline{u}]\|^2_{1,\sigma,H} + \sigma\|\overline{u}_{tt}\|^2_{0,\sigma,H}.\end{aligned} \tag{5.8}$$

Combining (5.3), (5.7) and (5.8) we obtain

$$\begin{aligned}&\sigma\|[\alpha-\acute{\alpha}, (\partial_t + \theta\cdot\nabla)(\alpha-\acute{\alpha}), \Omega_{lm}(\alpha-\acute{\alpha})]\|^2_{1,\sigma,C}\\ &\qquad \preccurlyeq \|[\overline{a}, \overline{b}, D\overline{a}, D\overline{b}]\|^2_{1,\sigma,Q} + \sigma(\|\overline{u}\|^2_{2,\sigma,H} + \|\overline{u}_t\|_{1,\sigma,H} + \|\overline{u}_{tt}\|_{0,\sigma,H}),\end{aligned} \tag{5.9}$$

for large enough σ, where the $\|\overline{u}, D\overline{u}\|_{0,\sigma,Q}$ term on the RHS of (5.7) is absorbed by the $\sigma\|\overline{u}\|_{1,\sigma,Q}$ term on the LHS of (5.3).

Below, we will need the observations that $\alpha - \alpha' = 0$ in a neighborhood of $\{\xi_i\}\times(-\infty, T]$ and α and its derivatives are bounded on the supports of $\overline{a}, \overline{b}$. Further, α is positive and bounded away from zero on $(T+1)\overline{B}\times(-\infty, T]$.

We use (5.9) and the relation between $\alpha-\acute{\alpha}$ and $\overline{a}+\theta\cdot\overline{b}$ to obtain estimates for $\overline{a}+\theta\cdot\overline{b}$. Using (1.9), we observe

$$\begin{aligned}\left(\partial_t + \theta\cdot\nabla\right)(\alpha-\acute{\alpha}) &= (a + \theta\cdot b)\alpha - (\acute{a} + \theta\cdot\acute{b})\acute{\alpha} - r^{-1}(\alpha-\acute{\alpha})\\ &= \alpha(\overline{a}+\theta\cdot\overline{b}) + (a + \theta\cdot b - r^{-1})(\alpha-\acute{\alpha}).\end{aligned} \tag{5.10}$$

This implies

$$\|\overline{a}+\theta\cdot\overline{b}\|^2_{0,\sigma,C} \preccurlyeq \|(\partial_t+\theta\cdot\nabla)(\alpha-\acute{\alpha})\|^2_{0,\sigma,C} + \|\alpha-\acute{\alpha}\|^2_{0,\sigma,C}. \tag{5.11}$$

Next, differentiating (5.10) w.r.t $\partial_t + \theta\cdot\nabla$ we obtain

$$\begin{aligned}(\partial_t+\theta\cdot\nabla)^2(\alpha-\acute{\alpha}) = \alpha(\partial_t+\theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b}) + f\,(\overline{a}+\theta\cdot\overline{b}) + g\,(\partial_t+\theta\cdot\nabla)(\alpha-\acute{\alpha})\\ + h\,(\alpha-\acute{\alpha})\end{aligned} \tag{5.12}$$

for some bounded functions f, g, h. Similarly, differentiating (5.10) w.r.t Ω_{lm} we obtain

$$\Omega_{lm}(\partial_t+\theta\cdot\nabla)(\alpha-\acute{\alpha}) = \alpha\,\Omega_{lm}(\overline{a}+\theta\cdot\overline{b}) + f\,(\overline{a}+\theta\cdot\overline{b}) + g\,\Omega_{lm}(\alpha-\acute{\alpha}) + h\,(\alpha-\acute{\alpha}) \tag{5.13}$$

for some bounded functions f, g, h.

Using (5.12), (5.13), we obtain

$$\|(\partial_t+\theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,C}+\sum_{l,m=1}^{3}\|\Omega_{lm}(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,C}$$
$$\preccurlyeq \|\overline{a}+\theta\cdot\overline{b}\|^2_{0,\sigma,C}+\|(\partial_t+\theta\cdot\nabla)\left(\alpha-\acute{\alpha}\right)\|^2_{1,\sigma,C}+\sum_{l,m=1}^{3}\|\Omega_{lm}\left(\alpha-\acute{\alpha}\right)\|^2_{1,\sigma,C}+\|\alpha-\acute{\alpha}\|^2_{1,\sigma,C}.$$

If we use (5.11) in this, we obtain

$$\|\overline{a}+\theta\cdot\overline{b}\|^2_{0,\sigma,C}+\|(\partial_t+\theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,C}+\sum_{l,m=1}^{3}\|\Omega_{lm}(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,C}$$
$$\preccurlyeq \|(\partial_t+\theta\cdot\nabla)\left(\alpha-\acute{\alpha}\right)\|^2_{1,\sigma,C}+\sum_{l,m=1}^{3}\|\Omega_{lm}\left(\alpha-\acute{\alpha}\right)\|^2_{1,\sigma,C}+\|\alpha-\acute{\alpha}\|^2_{1,\sigma,C}. \tag{5.14}$$

So combining (5.9) and (5.14) we obtain

$$\|\overline{a}+\theta\cdot\overline{b}\|^2_{0,\sigma,C}+\|(\partial_t+\theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,C}+\sum_{l,m=1}^{3}\|\Omega_{lm}(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,C}$$
$$\preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|_{0,\sigma,Q}+\|\overline{u}\|^2_{2,\sigma,H}+\|\overline{u}_t\|_{1,\sigma,H}+\|\overline{u}_{tt}\|_{0,\sigma,H}. \tag{5.15}$$

We now obtain estimates of a derivative of $\overline{a}+\theta\cdot\overline{b}$ in a direction not tangential to C, using the V solution. We note that $\overline{v}:=v-\acute{v}$ is a solution of the characteristic BVP

$$\mathcal{L}\overline{v}=2\overline{a}\acute{v}_t-2\overline{b}\cdot\nabla\acute{v}+\Big((b+\acute{b})\overline{b}-(a+\acute{a})\overline{a}\Big)\acute{v}, \quad \text{in } Q, \tag{5.16}$$

$$2\left(\partial_t+\theta\cdot\nabla-(\overline{a}+\theta\cdot\overline{b})+r^{-1}\right)\overline{v}=\acute{\mathcal{L}}\acute{\alpha}-\mathcal{L}\alpha+2(\overline{a}+\theta\cdot\overline{b})\acute{v}, \quad \text{on } C. \tag{5.17}$$

So applying Proposition 1.8 to $\overline{v}$ on the region Q and using (5.16), we obtain

$$\sigma\|\overline{v}\|^2_{1,\sigma,C}\preccurlyeq \|[\overline{a},\overline{b}]\|^2_{0,\sigma,Q}+\sigma\|\overline{v}\|^2_{1,\sigma,H}+\sigma\|\overline{v}_t\|^2_{0,\sigma,H},$$

so, in particular,

$$\|\overline{v}\|^2_{0,\sigma,C}+\|(\partial_t+\theta\cdot\nabla)\overline{v}\|^2_{0,\sigma,C}\preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b}]\|^2_{0,\sigma,Q}+\|\overline{v}\|^2_{1,\sigma,H}+\|\partial_t\overline{v}\|^2_{0,\sigma,H}. \tag{5.18}$$

From (5.17) we have

$$|(\partial_t+\theta\cdot\nabla)\overline{v}|+|\overline{v}|\succeq|\acute{\mathcal{L}}\acute{\alpha}-\mathcal{L}\alpha|-|[\overline{a},\overline{b}]|, \qquad \text{on } C.$$

So using this in (5.18) we obtain

$$\|\mathcal{L}\alpha-\acute{\mathcal{L}}\acute{\alpha}\|^2_{0,\sigma,C}\preccurlyeq\|[\overline{a},\overline{b}]\|^2_{0,\sigma,C}+\frac{1}{\sigma}\|[\overline{a},\overline{b}]\|^2_{0,\sigma,Q}+\|\overline{v}\|^2_{1,\sigma,H}+\|\partial_t\overline{v}\|^2_{0,\sigma,H}. \tag{5.19}$$

We introduce

$$\Delta_S := \frac{1}{2r^2}\sum_{l,m=1}^{3}\Omega_{lm}^2, \quad b^{\perp} := b - (b\cdot\theta)\theta$$

and claim (proof at the end of this section) that

$$\mathcal{L}\alpha = \alpha\left(\partial_t - \theta\cdot\nabla\right)(a+\theta\cdot b) - \Delta_S\alpha + 2b^{\perp}\cdot\nabla\alpha - (|b^{\perp}|^2 + 2r^{-1}\theta\cdot b)\alpha, \tag{5.20}$$

with a similar identity for $\acute{\mathcal{L}}$. What is significant here is that $\mathcal{L}\alpha$ has been expressed in terms of α and derivatives of α in directions tangential to C. The only derivative not tangential to C is applied to $a+\theta\cdot b$ and that is beneficial because it will be used to estimate this non-tangential derivative of $a+\theta\cdot b$.

Subtracting the identities for $\mathcal{L}\alpha$ and $\acute{\mathcal{L}}\acute{\alpha}$, we obtain

$$\begin{aligned}\mathcal{L}\alpha - \acute{\mathcal{L}}\acute{\alpha} = {} & \alpha(\partial_t - \theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b}) + (\alpha-\acute{\alpha})(\partial_t-\theta\cdot\nabla)(\acute{a}+\theta\cdot\acute{b}) - \Delta_S(\alpha-\acute{\alpha})\\ & + 2(b^{\perp}\cdot\nabla)(\alpha-\alpha') - 2\overline{b}^{\perp}\cdot\nabla\acute{\alpha} - (|b^{\perp}|^2 + 2r^{-1}\theta\cdot b)(\alpha-\acute{\alpha})\\ & - ((b^{\perp}+\acute{b}^{\perp})\cdot\overline{b}^{\perp} + 2r^{-1}\theta\cdot\overline{b})\acute{\alpha},\end{aligned}$$

implying

$$|\mathcal{L}\alpha - \acute{\mathcal{L}}\acute{\alpha}| \succeq |(\partial_t-\theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b})| - |\alpha-\acute{\alpha}| - |(b^{\perp}\cdot\nabla)(\alpha-\acute{\alpha})| - |\Delta_S(\alpha-\acute{\alpha})| - |\overline{b}|,$$

where we have used the fact that α has a positive lower bound on C. Hence using this in (5.19) we obtain

$$\begin{aligned}\|(\partial_t-\theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b})\|_{0,\sigma,C}^2 \preccurlyeq {} & \frac{1}{\sigma}\|[\overline{a},\overline{b}]\|_{0,\sigma,Q}^2 + \|[\overline{a},\overline{b}]\|_{0,\sigma,C}^2 + \|\overline{v}\|_{1,\sigma,H}^2 + \|\overline{v}_t\|_{0,\sigma,H}^2\\ & + \|[\alpha-\acute{\alpha}, (b^{\perp}\cdot\nabla)(\alpha-\acute{\alpha}), \Delta_S(\alpha-\acute{\alpha})]\|_{0,\sigma,C}^2.\end{aligned} \tag{5.21}$$

Now we combine the estimates obtained from the U and the V solutions. Now $b^{\perp} := b-(\theta\cdot b)\theta$ is perpendicular to θ (the radial direction) and has no component in the t axis direction, so $b^{\perp}\cdot\nabla$ is in the span of the Ω_{lm}. Further, Δ_S is a second order operator made up of Ω_{lm}. Hence

$$\|[(b^{\perp}\cdot\nabla)(\alpha-\acute{\alpha}), \Delta_S(\alpha-\acute{\alpha})]\|_{0,\sigma,C} \preccurlyeq \sum_{l,m=1}^{3}\|\Omega_{lm}(\alpha-\acute{\alpha})\|_{1,\sigma,C},$$

so using (5.9) in (5.21) we obtain

$$\begin{aligned}\|(\partial_t-\theta\cdot\nabla)(\overline{a}+\theta\cdot\overline{b})\|_{0,\sigma,C}^2 \preccurlyeq {} & \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|_{0,\sigma,Q}^2 + \|[\overline{a},\overline{b}]\|_{0,\sigma,C}^2 + \|\overline{u}\|_{2,\sigma,H}^2\\ & + \|[\overline{u}_t,\overline{v}]\|_{1,\sigma,H}^2 + \|[\overline{u}_{tt},\overline{v}_t]\|_{0,\sigma,H}^2.\end{aligned} \tag{5.22}$$

Now $\theta\cdot\nabla$ represent the radial derivative ∂_r in $\mathbb{R}^3$. Further $\partial_t-\partial_r, \partial_t+\partial_r, \Omega_{lm}$, $l,m=1,2,3$, span the tangent space to $\mathbb{R}^4$. Hence (5.15) and (5.22) give us

$$\begin{aligned}\|D(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,C} \preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|^2_{0,\sigma,Q} + \|[\overline{a},\overline{b}]\|^2_{0,\sigma,C} + \|\overline{u}\|^2_{2,\sigma,H} \\ + \|[\overline{u}_t,\overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt},\overline{v}_t]\|^2_{0,\sigma,H}. \end{aligned} \tag{5.23}$$

Also, from (5.15), we can extract

$$\|\overline{a}+\theta\cdot\overline{b}\|^2_{0,\sigma,C} \preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|^2_{0,\sigma,Q} + \|\overline{u}\|^2_{2,\sigma,H} + \|[\overline{u}_t,\overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt},\overline{v}_t]\|^2_{0,\sigma,H}. \tag{5.24}$$

Note that $\|[\overline{a},\overline{b}]\|_{0,\sigma,C}$ term is absent from the RHS of (5.24). This will be significant.

We integrate the last two inequalities w.r.t τ, over the interval $[\tau_{min},\tau_{max}]$. Using integral relations similar to those used in the proof of Theorem 1.3, we obtain

$$\begin{aligned}\|D(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|_{0,\sigma,\mathbb{R}^3\times[0,T]} + \|[\overline{a},\overline{b}]\|_{0,\sigma,\mathbb{R}^3\times[0,T]} \\ + \int_{\tau_{min}}^{\tau_{max}} \left(\|\overline{u}\|^2_{2,\sigma,H} + \|[\overline{u}_t,\overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt},\overline{v}_t]\|^2_{0,\sigma,H}\right)\, d\tau \end{aligned} \tag{5.25}$$

and

$$\begin{aligned}\|\overline{a}+\theta\cdot\overline{b}\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|_{0,\sigma,\mathbb{R}^3\times[0,T]} \\ + \int_{\tau_{min}}^{\tau_{max}} \left(\|\overline{u}\|^2_{2,\sigma,H} + \|[\overline{u}_t,\overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt},\overline{v}_t]\|^2_{0,\sigma,H}\right)\, d\tau. \end{aligned} \tag{5.26}$$

We have (5.26) for $\xi=\xi_i$, $i=1,2,3,4$ and the locations $\xi_1,\cdots,\xi_4$ are diverse with respect to $(T+1)B$, hence

$$\begin{aligned}\|[\overline{a},\overline{b}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|_{0,\sigma,\mathbb{R}^3\times[0,T]} \\ + \int_{\tau_{min}}^{\tau_{max}} \sum_{i=1}^{4} \left(\|\overline{u}\|^2_{2,\sigma,H} + \|[\overline{u}_t,\overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt},\overline{v}_t]\|^2_{0,\sigma,H}\right)\, d\tau. \end{aligned} \tag{5.27}$$

Using this in (5.25) we obtain (for each ξ_i)

$$\begin{aligned}\|[\overline{a},\overline{b}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} + \|D(\overline{a}+\theta\cdot\overline{b})\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \preccurlyeq \frac{1}{\sigma}\|[\overline{a},\overline{b},D\overline{a},D\overline{b}]\|_{0,\sigma,\mathbb{R}^3\times[0,T]} \\ + \int_{\tau_{min}}^{\tau_{max}} \sum_{i=1}^{4} \left(\|\overline{u}\|^2_{2,\sigma,H} + \|[\overline{u}_t,\overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt},\overline{v}_t]\|^2_{0,\sigma,H}\right)\, d\tau. \end{aligned} \tag{5.28}$$

Now

$$|D\overline{a}+\theta\cdot D\overline{b}| \preccurlyeq |[\overline{a},\overline{b}]| + |D(\overline{a}+\theta\cdot\overline{b})|$$

and the locations $\xi_1, \cdots, \xi_4$ are diverse with respect to $(T+1)B$, hence

$$\|[\overline{a}, \overline{b}, D\overline{a}, D\overline{b}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \preccurlyeq \frac{1}{\sigma}\|[\overline{a}, \overline{b}, D\overline{a}, D\overline{b}]\|_{0,\sigma,\mathbb{R}^3\times[0,T]} + \int_{\tau_{min}}^{\tau_{max}} \sum_{i=1}^{4} \left(\|\overline{u}\|^2_{2,\sigma,H} + \|[\overline{u}_t, \overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt}, \overline{v}_t]\|^2_{0,\sigma,H}\right) d\tau.$$

so taking σ large enough, we obtain

$$\|[\overline{a}, \overline{b}, D\overline{a}, D\overline{b}]\|^2_{0,\sigma,\mathbb{R}^3\times[0,T]} \preccurlyeq \int_{\tau_{min}}^{\tau_{max}} \sum_{i=1}^{4} \left(\|\overline{u}\|^2_{2,\sigma,H} + \|[\overline{u}_t, \overline{v}]\|^2_{1,\sigma,H} + \|[\overline{u}_{tt}, \overline{v}_t]\|^2_{0,\sigma,H}\right) d\tau.$$

Now $\overline{c} = \overline{a}_t - \nabla \cdot \overline{b}$, so the proof of Theorem 1.7 is complete except for a verification of the claim (5.20).

We verify the claim (5.20) when $\xi = 0$ as the general case follows by translation. We have

$$\begin{aligned}
\mathcal{L}\alpha &= \left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \Delta_S - 2a\partial_t + 2b\cdot\nabla + a^2 - |b|^2\right)\alpha \\
&= \frac{1}{r}\left(r\partial_t^2 - r\partial_r^2 - 2\partial_r\right)\alpha - \left(\Delta_S + 2a\partial_t - 2b\cdot\nabla - a^2 + |b|^2\right)\alpha \\
&= \frac{1}{r}\left(\partial_t^2 - \partial_r^2\right)(r\alpha) - \left(\Delta_S + 2a\partial_t - 2b\cdot\nabla - a^2 + |b|^2\right)\alpha \\
&= \frac{1}{r}\left(\partial_t - \partial_r\right)\left(\left(\partial_t + \partial_r\right)r\alpha\right) - \left(\Delta_S + 2a\partial_t - 2b\cdot\nabla - a^2 + |b|^2\right)\alpha \qquad (5.29) \\
&= \frac{1}{r}\left(\partial_t - \partial_r\right)\left(r\alpha(a + \theta\cdot b)\right) - \left(\Delta_S + 2a\partial_t - 2b\cdot\nabla - a^2 + |b|^2\right)\alpha \\
&= \alpha(\partial_t - \partial_r)(a + \theta\cdot b) - \frac{(a+\theta\cdot b)\alpha}{r} + (a+\theta\cdot b)(\alpha_t - \alpha_r) \\
&\quad - \left(\Delta_S + 2a\partial_t - 2b\cdot\nabla - a^2 + |b|^2\right)\alpha;
\end{aligned}$$

we used (1.9) in (5.29). As a consequence, we have

$$\begin{aligned}
&\mathcal{L}\alpha - \alpha(\partial_t - \partial_r)(a + \theta\cdot b) + \Delta_S\alpha \\
&= -\frac{(a+\theta\cdot b)\alpha}{r} + (a+\theta\cdot b)(\alpha_t - \alpha_r) - 2a\alpha_t + 2((\theta\cdot b)\theta + b^\perp)\cdot\nabla\alpha + \left(a^2 - |b|^2\right)\alpha \\
&= -\frac{(a+\theta\cdot b)\alpha}{r} + (a+\theta\cdot b)(\alpha_t - \alpha_r) - 2a\alpha_t + 2(\theta\cdot b)\alpha_r + 2b^\perp\cdot\nabla\alpha + \left(a^2 - |b|^2\right)\alpha \\
&= -\frac{(a+\theta\cdot b)\alpha}{r} - (a-\theta\cdot b)(\alpha_t + \alpha_r) + 2b^\perp\cdot\nabla\alpha + \left(a^2 - |b|^2\right)\alpha \qquad (5.30) \\
&= -\frac{(a+\theta\cdot b)\alpha}{r} - (a-\theta\cdot b)(a + \theta\cdot b - r^{-1})\alpha + 2b^\perp\cdot\nabla\alpha + \left(a^2 - |b|^2\right)\alpha \\
&= 2b^\perp\cdot\nabla\alpha - 2r^{-1}(\theta\cdot b)\alpha - \left(a^2 - (\theta\cdot b)^2\right)\alpha + \left(a^2 - |b|^2\right)\alpha \\
&= 2b^\perp\cdot\nabla\alpha - 2r^{-1}(\theta\cdot b)\alpha - |b^\perp|^2\alpha,
\end{aligned}$$

where we used (1.9) in (5.30). This proves the claim (5.20).

6. The forward problems

We give proofs of Propositions 1.1 and 1.2 using the standard progressing wave expansion method; one has to go through the computations to be certain that everything works, particularly in a cylindrical neighborhood of $\{\xi\} \times \mathbb{R}$. It is enough to prove the proposition when $\tau = 0$, $\xi = 0$ since the general case follows from a translation argument. So a, b, q are compactly supported smooth functions on $\mathbb{R}^3 \times \mathbb{R}$ which are zero on $B_\epsilon \times \mathbb{R}$ for some $\epsilon > 0$. Here B_ϵ is the origin centered open ball of radius ϵ. Also, we write $U(x,t;0,0), V(x,t;0,0), \alpha(x,t;0)$ as $U(x,t), V(x,t), \alpha(x,t)$.

The uniqueness of the distributional solution follows from the proof of uniqueness for Proposition 9.3. It remains to prove the existence and the structure of U, V.

We recall

$$\mathcal{M} = -2a\partial_t + 2b\cdot\nabla + q, \quad \mathcal{T} = \partial_t + \theta\cdot\nabla - (a + \theta\cdot b) + r^{-1}$$

and

$$\mathcal{L} := (\partial_t - a)^2 - (\nabla - b)^2 + c = \Box - 2a\partial_t + 2b\cdot\nabla + q = \Box + \mathcal{M}$$

Also, for $x \neq 0$ we define $r = |x|$ and $\theta = x/|x|$. We will need two observations, described next, in the proofs of Propositions 1.1 and 1.2.

For an arbitrary smooth function h on $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ and an arbitrary distribution F on $\mathbb{R}$, we claim

$$\mathcal{L}\left(h(x,t)F(t-|x|)\right) = 2\mathcal{T}(h)F'(t-|x|) + (\mathcal{L}h)\,F(t-|x|), \qquad x \neq 0. \tag{6.1}$$

We give its brief derivation. For $x \neq 0$, we have

$$(\partial_t - a)\left(h\,F(t-|x|)\right) = hF'(t-|x|) + F(t-|x|)(\partial_t - a)h,$$
$$(\partial_t - a)^2\left(h\,F(t-|x|)\right) = hF''(t-|x|) + 2F'(t-|x|)(\partial_t - a)h + F(t-|x|)(\partial_t - a)^2 h.$$

Similarly

$$(\nabla - b)(h(x,t)F(t-|x|)) = -\theta hF'(t-|x|) + F(t-|x|)(\nabla - b)h,$$
$$\begin{aligned}
&(\nabla - b)^2(h(x,t)F(t-|x|)) \\
&\quad = -(\nabla - b)\cdot(\theta hF'(t-|x|)) + (\nabla - b)\cdot(F(t-|x|)(\nabla - b)h) \\
&\quad = hF''(t-|x|) - F'(t-|x|)(\nabla - b)\cdot(\theta h) \\
&\qquad - \theta\cdot((\nabla - b)h)\,F'(t-|x|) + \left((\nabla - b)^2 h\right)F(t-|x|) \\
&\quad = hF''(t-|x|) - 2\left(\theta\cdot(\nabla - b)h + r^{-1}h\right)F'(t-|x|) + \left((\nabla - b)^2 h\right)F(t-|x|)
\end{aligned}$$

Hence (6.1) follows.

We will also need the solution of the transport equation

$$(\mathcal{T}f)(r\theta, t_0 + r) = g(r\theta, t_0 + r), \qquad r \neq 0. \tag{6.2}$$

We summarise the claim as the following lemma.

Lemma 6.1. *Suppose $g(x,t)$ is a smooth function on $\mathbb{R}^3 \times \mathbb{R}$ which is zero on $B_\epsilon \times \mathbb{R}$ and the restriction of g to the region $t \leq T$ is compactly supported for each T. Then (6.2) has a solution given by*

$$f(r\theta, r+t_0) = \alpha(r\theta, t_0+r) \int_0^r \frac{g(s\theta, t_0+s)}{\alpha(s\theta, t_0+s)}\, ds \tag{6.3}$$

with f smooth on $\mathbb{R}^3 \times \mathbb{R}$ and zero on $B_\epsilon \times \mathbb{R}$. Further, the restriction of f to $t \leq T$ is compactly supported with

$$\|f\|_{C^p(\mathbb{R}^3 \times (-\infty, T])} \leq C \|g\|_{C^p(\mathbb{R}^3 \times (-\infty, T])}, \tag{6.4}$$

with the constant C determined by ϵ, T and $\|[a,b]\|_{C^p(\mathbb{R}^3 \times (-\infty, T])}$.

We give the short proof of the lemma. For any smooth function $f(x,t)$ and any t_0 we have

$$\frac{d}{dr}(rf(r\theta, t_0+r)) = [r(f_t + \theta \cdot \nabla f) + f](r\theta, t_0+r), \qquad r \neq 0,$$

hence

$$r(\mathcal{T}f)(r\theta, t_0+r) = \frac{d}{dr}(rf(r\theta, t_0+r)) - (a + \theta \cdot b)(rf(r\theta, t_0+r)), \qquad r \neq 0. \tag{6.5}$$

Therefore (6.5) may be rewritten as the ODE

$$\frac{d}{dr}(rf(r\theta, t_0+r)) - [(a+\theta \cdot b)(rf)](r\theta, t_0+r)) = rg(r\theta, t_0+r), \qquad r \neq 0.$$

An integrating factor for this ODE is (note a, b are zero in $B_\epsilon \times \mathbb{R}$)

$$\begin{aligned}\exp\left(-\int_0^r (a+\theta \cdot b)(s\theta, t_0+s)\, ds\right) &= \exp\left(-\int_0^r (a+\theta\cdot b)((r-s)\theta, t_0+r-s)\, ds\right) \\ &= \frac{1}{r\alpha(r\theta, t_0+r)}\end{aligned}$$

so the ODE may be rewritten as

$$\frac{d}{dr}\left(\frac{f(r\theta, t_0+r)}{\alpha(r\theta, t_0+r)}\right) = \frac{g(r\theta, t_0+r)}{\alpha(r\theta, t_0+r)}, \qquad r \neq 0.$$

Hence, one solution of (6.2) is

$$f(r\theta, t_0+r) = \alpha(r\theta, t_0+r) \int_0^r \frac{g(s\theta, t_0+s)}{\alpha(s\theta, t_0+s)}\, ds, \qquad r \neq 0;$$

note that $f(x,t)$, is zero in $B_\epsilon \times \mathbb{R}$ and smooth on $\mathbb{R}^3 \times \mathbb{R}$. Further f is compactly supported when restricted to $t \leq T$ and

$$\|f\|_{C^p(\mathbb{R}^3 \times (-\infty, T])} \leq C \|g\|_{C^p(\mathbb{R}^3 \times (-\infty, T])},$$

with the constant C determined by ϵ, T and $\|[a,b]\|_{C^p(\mathbb{R}^3 \times (-\infty, T])}$.

6.1. **Proof of Proposition 1.1.** One may verify with a standard argument that

$$\Box \left(\frac{H(t-|x|)}{|x|} \right) = 4\pi H(t)\delta(x). \tag{6.6}$$

We seek a solution $U(x,t)$, of the IVP (1.2), (1.3), of the form

$$U(x,t) = \frac{H(t-|x|)}{|x|} + u(x,t)H(t-|x|),$$

with $u(x,t)$ a smooth function in the region $t \geq |x|$, satisfying $u(x,t) = 0$ in a neighborhood of $(x=0, t=0)$.

Clearly such a U satisfies (1.3). So we need to find such a $u(x,t)$ so that

$$\mathcal{L}(u(x,t)H(t-|x|)) = -\mathcal{M}\left(\frac{H(t-|x|)}{|x|} \right).$$

Since $\mathcal{M} = 0$, $a = 0, b = 0$ on $B_\epsilon \times \mathbb{R}$, we have

$$\mathcal{M}\left(\frac{H(t-|x|)}{|x|} \right) = \mathcal{M}(|x|^{-1})H(t-|x|) - 2|x|^{-1}\left(a + \theta \cdot b\right)\delta(t-|x|),$$

hence we want

$$\mathcal{L}[u(x,t)H(t-|x|)] = 2|x|^{-1}(a+\theta\cdot b)\,\delta(t-|x|) - \mathcal{M}(|x|^{-1})\,H(t-|x|). \tag{6.7}$$

For N large enough to be chosen later, we seek $u(x,t)H(t-|x|)$ in the form

$$u(x,t)H(t-|x|) = a_0(x,t)H(t-|x|) + \sum_{k=1}^{N} a_k(x,t)\frac{(t-|x|)_+^k}{k!} + S_N(x,t) \tag{6.8}$$

for suitably chosen smooth functions a_k which will be zero on $B_\epsilon \times \mathbb{R}$, and a S_N which will be highly differentiable as N increases, zero in a neighborhood of $(0,0)$ and supported in $t \geq |x|$. With such an expansion we will define

$$u(x,t) := \sum_{k=0}^{N} a_k(x,t)\frac{(t-|x|)_+^k}{k!} + S_N(x,t). \tag{6.9}$$

We construct the a_k and S_N so that (6.7) holds. Since the a_k are to be zero on $B_\epsilon \times \mathbb{R}$, it is clear that

$$\mathcal{L}\left(a_0(x,t)H(t-|x|) + \sum_{k=1}^{N} a_k(x,t)\frac{(t-|x|)_+^k}{k!} \right) = 0, \qquad \text{on } B_\epsilon \times \mathbb{R}.$$

Further, for $x \neq 0$, noting that

$$H(s) = s_+^0, \qquad \frac{d}{ds}\delta(s) = \delta'(s), \qquad \frac{d}{ds}H(s) = \delta(s), \qquad \frac{d}{ds}\frac{s_+^k}{k!} = \frac{s_+^{k-1}}{(k-1)!}, \qquad k \geq 1,$$

and using (6.1) we obtain (for $x \neq 0$)

$$\begin{aligned}
&\mathcal{L}\left(a_0(x,t)H(t-|x|) + \sum_{k=1}^{N} a_k(x,t)\frac{(t-|x|)_+^k}{k!} \right) \\
&\quad = 2(\mathcal{T}a_0)\delta(t-|x|) + (\mathcal{L}a_0)H(t-|x|) + \sum_{k=1}^{N} 2(\mathcal{T}a_k)\frac{(t-|x|)_+^{k-1}}{(k-1)!} + (\mathcal{L}a_k)\frac{(t-|x|)_+^k}{k!} \\
&\quad = 2(\mathcal{T}a_0)\delta(t-|x|) + \sum_{k=1}^{N}(2\mathcal{T}a_k + \mathcal{L}a_{k-1})\frac{(t-|x|)_+^{k-1}}{(k-1)!} + \mathcal{L}a_N\frac{(t-|x|)_+^N}{N!}.
\end{aligned}$$

Keeping in mind (6.7), we choose $a_0(x,t)$ and $a_1(x,t)$ so that

$$2\mathcal{T}a_0 = 2|x|^{-1}(a+\theta\cdot b), \qquad \text{on } x \neq 0 \tag{6.10}$$

$$2\mathcal{T}a_1 + \mathcal{L}a_0 = -\mathcal{M}(|x|^{-1}), \qquad \text{on } x \neq 0, \tag{6.11}$$

and, for $2 \leq k \leq N$, we choose a_k so that

$$2\mathcal{T}a_k + \mathcal{L}(a_{k-1}) = 0, \qquad \text{on } x \neq 0. \tag{6.12}$$

Assuming for the moment that we have constructed smooth a_k satisfying these equations with a_k zero on $B_\epsilon \times \mathbb{R}$, keeping in mind (6.7), we need to find S_N which solves

$$\mathcal{L}S_N = -\left(\mathcal{L}a_N\right)\frac{(t-|x|)_+^N}{N!}, \qquad \text{on } \mathbb{R}^3\times\mathbb{R}, \tag{6.13}$$

$$S_N = 0, \qquad \text{on } t < 0. \tag{6.14}$$

Since a_N is a smooth function that is zero in $B_\epsilon \times \mathbb{R}$, the function $\mathcal{L}a_N\frac{(t-|x|)_+^N}{N!}$ is in $C^{N-1}(\mathbb{R}^3 \times \mathbb{R})$, zero in a neighborhood of $(0,0)$ and supported in the region $t \geq |x|$. Hence, if $N > 5$, then by Proposition 9.3 with $m = N-1$ the IVP (6.13), (6.14) has a unique distributional solution which is in $C^{N-3}(\mathbb{R}^3 \times \mathbb{R})$. Further

$$\|S_N\|_{C^{N-3}(\mathbb{R}^3\times(-\infty,T])} \leq C\|\mathcal{L}a_N\|_{C^{N-1}(\mathbb{R}^3\times(-\infty,T])}., \tag{6.15}$$

with C determined by T and $\|[a,b,q]\|_{C^{N-1}(\mathbb{R}^3\times(-\infty,T])}$.

Hence S_N is at least C^2, so by a standard energy estimate argument, one can show that S_N is supported in the region $t \geq |x|$ and $S_N = 0$ in a neighborhood of $(0,0)$. Hence, if we take $N > 5$ then the u defined by (6.9) is in $C^{N-3}(\mathbb{R}^3 \times \mathbb{R})$, zero in a neighborhood of $(0,0)$ and $u(x,t)H(t-|x|)$ is the (unique) distributional solution of the IVP (1.11), (1.12). Now N was arbitrary and u is uniquely determined on $t \geq |x|$, hence u is smooth on $t \geq |x|$. Since (6.7) holds, we see that (1.13) holds.

If remains to prove that there are smooth $a_k(x,t)$ which satisfy (6.10), (6.11), (6.12), are zero in $B_\epsilon \times \mathbb{R}$, that for the u defined by (6.9) the relation (1.14) holds, and we have the estimate on $\|u\|_{C^3(Q_{0,0})}$ claimed in Proposition 1.1.

Since the RHS (6.10) is smooth, compactly supported and zero on $B_\epsilon \times \mathbb{R}$, from Lemma 6.1, we can construct a smooth a_0 satisfying (6.10), which is zero on $B_\epsilon \times \mathbb{R}$ and its restriction to $t \leq T$ is compactly supported. Further

$$\|a_0\|_{C^p(\mathbb{R}^3 \times (-\infty, T])} \leq C$$

with C dependent only on ϵ, T and $\|[a,b]\|_{C^p(\mathbb{R}^3 \times (-\infty,T])}$. Next, rewriting (6.11) as

$$\mathcal{T} a_1 = -\frac{1}{2}(\mathcal{L} a_0 + \mathcal{M}(|x|^{-1})), \qquad x \neq 0,$$

again, from Lemma 6.1, there is a smooth solution a_1 of (6.11), which is zero on $B_\epsilon \times \mathbb{R}$ and its restriction to $t \leq T$ is compactly supported. Further, using the estimate on a_0

$$\begin{aligned} \|a_1\|_{C^p(\mathbb{R}^3 \times (-\infty,T])} &\leq C \|\mathcal{L} a_0 + \mathcal{M}(|x|^{-1})\|_{C^p(\mathbb{R}^3 \times (-\infty,T])} \\ &\leq C(\|a_0\|_{C^{p+2}(\mathbb{R}^3 \times (-\infty,T])} + \|[a,b,q]\|_{C^p(\mathbb{R}^3 \times (-\infty,T])}) \\ &\leq C_1 \end{aligned}$$

where C_1 is a constant determined by ϵ, T and $\|[a,b,q]\|_{C^{p+2}(\mathbb{R}^3 \times (-\infty,T])}$. Next, for any $2 \leq k \leq N$ we may write (6.12) as

$$\mathcal{T} a_k = -\frac{1}{2} \mathcal{L} a_{k-1}, \qquad x \neq 0,$$

hence, from Lemma 6.1, there is a smooth solution a_k of (6.12) which is zero on $B_\epsilon \times \mathbb{R}$ and its restriction to $t \leq T$ is compactly supported. Further

$$\|a_k\|_{C^p(\mathbb{R}^3 \times (-\infty,T])} \leq C \|\mathcal{L} a_{k-1}\|_{C^p(\mathbb{R}^3 \times (-\infty,T])} \leq C_2 \|a_{k-1}\|_{C^{p+2}(\mathbb{R}^3 \times (-\infty,T])}.$$

with C_2 determined by ϵ, T and $\|[a,b,q]\|_{C^p(\mathbb{R}^3 \times (-\infty,T])}$. So by induction,

$$\|a_k\|_{C^p(\mathbb{R}^3 \times (-\infty,T])} \leq C$$

with C determined by ϵ, T, k and $\|[a,b,q]\|_{C^{p+2k}(\mathbb{R}^3 \times (-\infty,T])}$ for $0 \leq k \leq N$. In particular

$$\|a_N\|_{C^p(\mathbb{R}^3 \times (-\infty,T])} \leq C$$

with C determined by ϵ, T, N and $\|[a,b,q]\|_{C^{p+2N}(\mathbb{R}^3 \times (-\infty,T])}$. We use this estimate in (6.15) to obtain

$$\|S_N\|_{C^{N-3}(\mathbb{R}^3 \times (-\infty,T])} \leq C \|a_N\|_{C^{N+1}(\mathbb{R}^3 \times (-\infty,T])} \leq C_1$$

with C_1 determined by ϵ, T and $\|[a,b,q]\|_{C^{3N+1}(\mathbb{R}^3 \times (-\infty,T]}$. In particular, taking $N = 6$ we have

$$\|S_6\|_{C^3(\mathbb{R}^3 \times (-\infty,T])} \leq C$$

where C is determined by ϵ, T and $\|[a,b,q]\|_{C^{19}(\mathbb{R}^3 \times (-\infty,T]}$.

From the uniqueness of the distributional solution, we know that the u defined by (6.9) is independent of N on the region $t \geq |x|$, hence using the estimates on $a_0, a_1, \cdots, a_6$ and S_6 we have

$$\|u\|_{C^3(Q_{0,0})} \leq C$$

with C determined by ϵ, T and $\|[a,b,q]\|_{C^{19}(\mathbb{R}^3 \times (-\infty,T]}$. Here $Q_{0.0}$ is $\{(x,t) : |x| \leq t \leq T\}$.

Finally, since S_6 is in C^3 and supported on $t \geq |x|$, we observe from (6.9) that

$$u(x,|x|) = a_0(x,|x|), \qquad x \in \mathbb{R}^3.$$

From Lemma 6.1 applied to (6.10), and taking $t_0 = 0$, we have

$$a_0(r\theta, r) = \alpha(r\theta, r) \int_0^r \frac{(a+\theta\cdot b)(s\theta,s)}{s\alpha(s\theta,s)}\, ds.$$

Now, from (1.8)

$$\begin{aligned}\frac{(a+\theta\cdot b)(s\theta,s)}{s\alpha(s\theta,s)} &= (a+\theta\cdot b)(s\theta,s)\, \exp\left(-\int_0^s (a+\theta\cdot b)((s-\rho)\theta, s-\rho)\, d\rho\right)\\ &= (a+\theta\cdot b)(s\theta,s)\, \exp\left(-\int_0^s (a+\theta\cdot b)(\rho\theta,\rho)\, d\rho\right)\\ &= -\frac{d}{ds}\left[\exp\left(-\int_0^s (a+\theta\cdot b)(\rho\theta,\rho)\, d\rho\right)\right]\\ &= -\frac{d}{ds}\left(\frac{1}{s\alpha(s\theta,s)}\right).\end{aligned}$$

Noting that $\lim_{r\to 0^+}(r\alpha(r\theta,r)) = 1$, we obtain

$$a_0(r\theta,r) = \alpha(r\theta,r) - r^{-1},$$

proving (1.14).

6.2. **Proof of Proposition 1.2.** We have $\xi = 0, \tau = 0$ and a, b, c are zero in $B_\epsilon \times \mathbb{R}$. We seek a solution of the IVP (1.15) - (1.16) in the form

$$V(x,t) = |x|^{-1}\delta(t-|x|) + f(x,t)\delta(t-|x|) + v(x,t)H(t-|x|), \tag{6.16}$$

with $v(x,t)$ a smooth function in $t \geq |x|$, $f(x,t)$ a smooth function on $\mathbb{R}^3 \times \mathbb{R}$, $v(x,t)$ zero in a neighborhood of (x=0, t=0) and $f(x,t)$ zero on $B_\epsilon \times \mathbb{R}$. Clearly such a $V(x,t)$ will satisfy the initial condition (1.16), so we just need to find a solution of this form for (1.15).

Since

$$\square(|x|^{-1}\delta(t-|x|)) = 4\pi\delta(x)\delta(t)$$

we have (note $\mathcal{M}, a, b$ are zero in $B_\epsilon \times \mathbb{R}$)

$$\mathcal{L}(|x|^{-1}\delta(t-|x|)) = \mathcal{M}(|x|^{-1}\delta(t-|x|)) = \mathcal{M}(|x|^{-1})\delta(t-|x|) - 2|x|^{-1}(a+\theta\cdot b)\,\delta'(t-|x|).$$

Hence using (6.1) (we assume $f = 0$ in $B_\epsilon \times \mathbb{R}$) we have

$$\begin{aligned}&\mathcal{L}(|x|^{-1}\delta(t-|x|) + f(x,t)\delta(t-|x|)) - 4\pi\delta(x)\delta(t)\\ &\qquad = 2[\mathcal{T}f - |x|^{-1}(a+\theta\cdot b)]\,\delta'(t-|x|) + [\mathcal{L}f + \mathcal{M}(|x|^{-1})]\delta(t-|x|).\end{aligned}$$

Since $|x|^{-1}(a+\theta\cdot b)$ is zero on $B_\epsilon\times\mathbb{R}$, from Lemma 6.1, we can find a smooth $f(x,t)$ which is zero on $B_\epsilon\times\mathbb{R}$ and $\mathcal{T}f=|x|^{-1}(a+\theta\cdot b)$. In fact, from Lemma 6.1, we have

$$\begin{aligned} f(r\theta,r+t_0) &= \alpha(r\theta,t_0+r)\int_0^r \frac{(a+\theta\cdot b)(s\theta,t_0+s)}{s\alpha(s\theta,t_0+s)}\,ds \\ &= \alpha(r\theta,t_0+r)-r^{-1}, \end{aligned}$$

by the calculation at the end of subsection 6.1. Hence

$$f(x,t)=\alpha(x,t)-|x|^{-1}.$$

Note that, from (1.8), we have $\alpha(x,t)-|x|^{-1}=0$ in $B_\epsilon\times\mathbb{R}$.

So, keeping in mind (6.16) and (1.15), we seek $v(x,t)$, a smooth function on $t\geq|x|$ which is zero near $(0,0)$ and

$$\mathcal{L}[v(x,t)H(t-|x|)]=[\mathcal{L}(\alpha(x,t)-|x|^{-1})+\mathcal{M}(|x|^{-1})]\,\delta(t-|x|). \tag{6.17}$$

We seek $v(x,t)H(t-|x|)$ in the form

$$v(x,t)H(t-|x|)=\sum_{k=0}^{N} b_k(x,t)\frac{(t-|x|)_+^k}{k!}+R_N(x,t), \tag{6.18}$$

for some large N, for smooth functions b_k which are zero in $B_\epsilon\times\mathbb{R}$ and for some regular enough function R_N which is supported in $t\geq|x|$ and zero in a neighborhood of $(0,0)$. Then we will take

$$v(x,t)=\sum_{k=0}^{N} b_k(x,t)\frac{(t-|x|)^k}{k!}+R_N(x,t), \qquad t\geq|x|. \tag{6.19}$$

As seen in the proof of Proposition 1.1, we have

$$\begin{aligned} &\mathcal{L}\left(\sum_{k=0}^{N} b_k(x,t)\frac{(t-|x|)_+^k}{k!}\right) \\ &\quad = 2(\mathcal{T}b_0)\delta(t-|x|)+\sum_{k=1}^{N}(2\mathcal{T}b_k+\mathcal{L}b_{k-1})\frac{(t-|x|)_+^{k-1}}{(k-1)!}+(\mathcal{L}b_N)\frac{(t-|x|)_+^N}{N!}. \end{aligned}$$

So keeping in mind (6.17), we seek b_k such that

$$\mathcal{T}b_0=\mathcal{L}(\alpha(x,t)-|x|^{-1})+\mathcal{M}(|x|^{-1}), \qquad x\neq 0, \tag{6.20}$$

$$\mathcal{T}b_k=-\frac{1}{2}\mathcal{L}b_{k-1}, \qquad x\neq 0,\ 1\leq k\leq N. \tag{6.21}$$

Since the RHS of (6.20) is smooth, zero on $B_\epsilon\times\mathbb{R}$ and its restriction to $t\leq T$ is compactly supported, Lemma 6.1 guarantees a smooth solution b_0 of (6.20) with b_0 zero on $B_\epsilon\times\mathbb{R}$ and its restriction to $t\leq T$ compactly supported. Further

$$\|b_0\|_{C^p(\mathbb{R}^3\times(-\infty,T])}\leq C$$

with C determined by ϵ, T and $\|[a,b,q]\|_{C^{p+2}(\mathbb{R}^3\times(-\infty,T])}$. Applying Lemma 6.1 recursively to (6.21) we conclude that, for $1 \le k \le N$, there is a smooth solution b_k of (6.21) with b_k zero on $B_\epsilon \times \mathbb{R}$ and its restriction to $t \le T$ compactly supported. Further

$$\|b_k\|_{C^p(\mathbb{R}^3\times(-\infty,T])} \le C\|b_{k-1}\|_{C^{p+2}(\mathbb{R}^3\times(-\infty,T])}, \qquad 1 \le k \le N,$$

with C determined by ϵ, T and $\|[a,b]\|_{C^{p+2}(\mathbb{R}^3\times(-\infty,T])}$. Hence, by an induction argument,

$$\|b_N\|_{C^p(\mathbb{R}^3\times(-\infty,T])} \le C$$

with C determined by ϵ, T and $\|[a,b,q]\|_{C^{p+2N+2}(\mathbb{R}^3\times(-\infty,T])}$.

With the b_k chosen above we have

$$\mathcal{L}\left(\sum_{k=0}^{N} b_k(x,t)\frac{(t-|x|)_+^k}{k!} + R_N(x,t)\right) = (\mathcal{L}b_N)\frac{(t-|x|)_+^N}{N!} + \mathcal{L}R_N.$$

Hence for (6.17) to hold, we need to find a R_N which is supported on $t \ge |x|$, zero in a neighborhood of $(0,0)$ and is the solution of the IVP

$$\begin{aligned} \mathcal{L}R_N = -(\mathcal{L}b_N)\frac{(t-|x|)_+^N}{N!}, &\qquad \text{on } \mathbb{R}^3\times\mathbb{R}, \\ R_N = 0, &\qquad \text{for } t << 0. \end{aligned}$$

Then repeating the argument used in the proof of Proposition 1.1, we can show that the v defined by (6.19) is smooth on $t \ge |x|$, zero in a neighborhood of $(0,0)$ and

$$\|v\|_{C^3(Q_{0,0})} \le C$$

with C determined by ϵ, T and $\|[a,b,q]\|_{C^{21}(\mathbb{R}^3\times(-\infty,T])}$.

Noting that (6.17) implies (1.17), it remains to verify (1.18). From (6.19), we see that for $t \ge |x|$ we have

$$v(x,t) = b_0(x,t) + b_1(x,t)(t-|x|) + \sum_{k=2}^{6} b_k(x,t)(t-|x|)^k/k! + R_6(x,t).$$

Since R_6 is supported in $t \ge |x|$ and is at least C^2, we see that $\mathcal{T}R_6 = 0$ on $t = |x|$. Further, on $t = |x|$

$$\mathcal{T}(b_1(x,t)(t-|x|)) = (\mathcal{T}b_1(x,t))(t-|x|) + b_1\mathcal{T}(t-|x|) = b_1(x,t)(\partial_t + \theta\cdot\nabla)(t-|x|) = 0.$$

Hence, noting that $\mathcal{L} = \Box + \mathcal{M}$ and that $\Box(|x|^{-1}) = 0$ for $x \ne 0$, on $t = |x|$ we have

$$(\mathcal{T}v)(x,t) = (\mathcal{T}b_0)(x,t) = \mathcal{L}(\alpha(x,t) - |x|^{-1}) + \mathcal{M}(|x|^{-1}) = \mathcal{L}\alpha)(x,t), \qquad x \ne 0.$$

7. Proof of Proposition 1.8

It is sufficient to prove the proposition when $\mathcal{L} = \Box$ since the lower order terms can be absorbed in the LHS of the inequality. This argument also shows that the constant in the inequality depends only on $T, |\xi|, |\tau|$ and $\|[a,b]\|_{C^1(Q_{\xi,\tau})}$, $\|c\|_{C^0(Q_{\xi,\tau})}$.

We prove the proposition when $\xi=0,\tau=0$. The general ξ,τ case follows by translation. For the $\xi=0,\tau=0$ case, we denote $Q_{\xi,\tau}, H_{\xi,\tau}$ and $C_{\xi,\tau}$ by Q, H, C.

Our proof uses Theorem A.7 of [6] and we keep the notation used there. We first observe that Theorem A.7 in [6] is valid for the weight $\phi(x,t)=t$. Even though it does not satisfy the strong pseudo-convexity criterion needed in Theorem A.7 of [6], it does satisfy (A.25) in [6] which is what is needed to obtain the Carleman estimate in Theorem A.7 of [6].

For our problem, $p(x,t,\xi,\tau) = -\tau^2+\xi^2$ and $\phi(x,t)=t$. Hence

$$\begin{aligned} A &= p(x,t,\xi,\tau) - \sigma^2 p(x,t,\nabla\phi,\phi_t) = -\tau^2 + |\xi|^2 + \sigma^2, \\ B &= \{p,\phi\} = p_\tau \phi_t = -2\tau \end{aligned}$$

implying $\{A,B\}=0$. So (A.25) holds if we consider g to be any positive constant and then choose $d>0$ accordingly. Consequently we have

$$\sigma \int_Q e^{2\sigma t}\left(|\nabla_{x,t} w|^2 + \sigma^2 w^2\right) + \sigma\int_{\partial Q} \nu\cdot E \le C\int_Q e^{2\sigma t}|\Box w|^2 \tag{7.1}$$

where $\nu=(\nu_0,\nu_1,\nu_2,\nu_3)$ is the outward unit normal and the zero index corresponds to t. Now $\partial Q = C\cup H$ and we compute the expressions appearing in the integral over C and H.

Using the calculation of the boundary terms for the wave operator from subsection A.2 in [6], we write

$$\begin{aligned} \frac{1}{2}E_j &= 2(e^{\sigma t}w)_{x_j}(e^{\sigma t}w)_t - g(x,t)(e^{\sigma t}w)_{x_j} e^{\sigma t} w, \quad j\in\{1,2,3\}, \\ &= e^{2\sigma t}\left(2\sigma w w_{x_j} + 2 w_{x_j} w_t - g w w_{x_j}\right) \end{aligned}$$

and,

$$\begin{aligned} \frac{1}{2}E_0 &= -|\nabla_{x,t}(e^{\sigma t}w)|^2 - \sigma^2 (e^{\sigma t}w)^2 + g e^{\sigma t} w (e^{\sigma t} w)_t \\ &= e^{2\sigma t}\left(-(w_t+\sigma w)^2 - |\nabla w|^2 - \sigma^2 w^2 + g w(w_t+\sigma w)\right) \\ &= e^{2\sigma t}\left(-|\nabla_{x,t} w|^2 - 2\sigma^2 w^2 - 2\sigma w w_t + g w (w_t + \sigma w)\right). \end{aligned}$$

On C, we have $\sqrt{2}\,\nu(x,t) = (-1,\theta)$, hence

$$\begin{aligned} \nu\cdot E = \sqrt{2}\sigma e^{2\sigma t}&\left[(|\nabla_{x,t}w|^2 + 2w_t\,\theta\cdot\nabla w) + 2\sigma w(w_t + \theta\cdot\nabla_x w) + 2\sigma^2 w^2 - \sigma g w^2 \right. \\ &\left. - g w\left(w_t + \theta\cdot\nabla w\right)\right]. \end{aligned} \tag{7.2}$$

Now

$$\begin{aligned} |\nabla_{x,t} w|^2 + 2 w_t\,\theta\cdot\nabla_x w &= w_t^2 + (\theta\cdot\nabla w)^2 + \sum_{i<j}\frac{(\Omega_{ij} v)^2}{|x|^2} + 2 w_t\,\theta\cdot\nabla w \\ &= \sum_{i<j}\frac{(\Omega_{ij}w)^2}{|x|^2} + (w_t + \theta\cdot\nabla w)^2 \end{aligned} \tag{7.3}$$

where $\Omega_{ij} = x^i\partial_j - x^j\partial_i$, $i,j = 1,2,3$ are the angular derivatives. If we define

$$P = w_t + \theta\cdot\nabla w$$

then using (7.3) in (7.2) we obtain

$$\begin{aligned}\nu\cdot E &= \sqrt{2}\sigma e^{2\sigma t}\left(|x|^{-2}\sum_{i<j}(\Omega_{ij}w)^2 + P^2 + 2\sigma Pw + 2\sigma^2w^2 - \sigma g w^2 - gPw\right)\\ &\geq \sqrt{2}\sigma e^{2\sigma t}\left(|x|^{-2}\sum_{i<j}(\Omega_{ij}w)^2 + P^2 - 2\sigma|P||w| + 2\sigma^2w^2 - \sigma\|g\|_\infty w^2 - \|g\|_\infty Pw\right)\\ &\geq \sqrt{2}\sigma e^{2\sigma t}\left(|x|^{-2}\sum_{i<j}(\Omega_{ij}w)^2 + P^2\left(1-\epsilon-\delta\right) + w^2\left(2\sigma^2 - \frac{\sigma^2}{\epsilon} - \sigma\|g\|_\infty - \frac{1}{4\delta}\|g\|_\infty^2\right)\right).\end{aligned}$$

Taking $\epsilon = \frac{3}{4}$, $\delta = \frac{1}{8}$ and $\sigma > 0$ large, we have

$$\nu\cdot E \succeq \sigma e^{2\sigma t}\left(|x|^{-2}\sum_{i<j}(\Omega_{ij}w)^2 + (w_t+\theta\cdot\nabla w)^2\right) + \sigma^2w^2, \qquad \text{on } C. \tag{7.4}$$

Next, on H, noting that $\nu = (1,0,0,0)$, using the A.M-G.M inequality as done in obtaining (7.4), we obtain

$$\begin{aligned}\nu\cdot E = E_0 &= 2e^{2\sigma t}\left(-|\nabla_{x,t}w|^2 - 2\sigma^2w^2 - 2\sigma w w_t + gw\left(w_t+\sigma w\right)\right)\\ &\succeq - e^{2\sigma t}\left(|\nabla_{x,t}w|^2 + \sigma^2w^2\right),\end{aligned}$$

hence

$$|\nu\cdot E| \preccurlyeq e^{2\sigma t}\left(|\nabla_{x,t}w|^2 + \sigma^2w^2\right) \qquad \text{on } H. \tag{7.5}$$

So using (7.4) and (7.5) in (7.1), we obtain

$$\begin{aligned}\sigma\int_Q e^{2\sigma t}\left(|\nabla_{x,t}w|^2+\sigma^2w^2\right) &+ \sigma\int_C e^{2\sigma t}\left(|\nabla_C w|^2+\sigma^2w^2\right)\\ &\preccurlyeq \int_Q e^{2\sigma t}|\Box w|^2 + \sigma\int_H e^{2\sigma t}\left(|\nabla_{x,t}w|^2+\sigma^2w^2\right).\end{aligned}$$

where the constant is independent of w, σ and depends only on τ, T. This completes the proof of the proposition.

8. Construction of a diverse set of locations

Let D be a non-empty bounded open subset of $\mathbb{R}^d$. We give two ways to construct a diverse set of locations with respect D. If $\xi_1,\cdots,\xi_k$ is a collection of vectors in $\mathbb{R}^d$ we define the "hyperplane" determined by these vectors as

$$\mathrm{hyp}(\xi_1, \cdots, \xi_k) := \left\{ \sum_{i=1}^{k} \alpha_i \xi^i : \alpha_i \in \mathbb{R},\ \sum_{i=1}^{k} \alpha_i = 1 \right\}$$

and the convex hull of $\xi_1, \cdots, \xi_k$ is defined as

$$\mathrm{conv}(\xi_1, \cdots, \xi_k) = \left\{ \sum_{i=1}^{k} \alpha_i \xi_i : \alpha_i \geq 0,\ \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

The following proposition gives two ways to generate a diverse collection of sources.

Proposition 8.1. *Suppose d is a positive integer and D is a non-empty bounded open subset of $\mathbb{R}^d$.*

- *(a) Suppose $\xi_1, \cdots, \xi_d$ are linearly independent vectors in $\mathbb{R}^d$ and $\xi_{d+1} \in conv(\xi_1, \cdots, \xi_d)$ but different from $\xi_1, \cdots, \xi_d$. If $hyp(\xi_1, \cdots, \xi_d)$ does not intersect $\overline{D}$ then $\xi_1, \cdots, \xi_{d+1}$ is a diverse set of locations with respect to D.*
- *(b) If $\xi_1, \cdots, \xi_{d+1}$ is a set of locations in $\mathbb{R}^d \setminus \overline{D}$ such that $\overline{D}$ is in the interior of $conv(\xi_1, \cdots, \xi_{d+1})$ then $\xi_1, \cdots, \xi_{d+1}$ is a diverse set of locations with respect to D.*

Remark. If $\rho > 0$ and $N > \rho\sqrt{d}$ then $Ne_1, \cdots, Ne_d, N(e_1 + \cdots + e_d)/d$ is a diverse set of locations collection with respect to ρB. This is so because of (a) and that $\mathrm{hyp}(Ne_1, \cdots, Ne_d)$ does not intersect $\rho\overline{B}$.

Proof. Suppose $\xi_i \in \mathbb{R}^d \setminus \overline{D}$, $i = 1, \cdots, d+1$. For any $x \in \overline{D}$, define

$$\theta_i(x) = \frac{x - \xi_i}{|x - \xi_i|}, \qquad i = 1, \cdots, d+1,$$

and the $(d+1) \times (d+1)$ matrix

$$M(x) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \theta_1(x) & \theta_1(x) & \cdots & \theta_{d+1}(x) \end{bmatrix}, \qquad x \in \overline{D}.$$

Then $\xi_1, \cdots, \xi_{d+1}$ is a diverse set of locations with respect to D iff

$$\|[a, b]\| \preccurlyeq \|M(x)[a, b]\|, \qquad \forall x \in \overline{D}, a \in \mathbb{R}, b \in \mathbb{R}^d,$$

with the constant independent of x, a, b. This condition is equivalent to the invertibility of $M(x)$ for all $x \in \overline{D}$ because the invertibility of $M(x)$ implies the operator norm $\|M(x)\|$ is positive so the continuous map

$$x \in \overline{D} \to M(x) \to \|M(x)\|$$

has a positive lower bound since $\overline{D}$ is compact.

Proof of (a). By hypothesis,

$$\xi_{d+1} = \sum_{i=1}^{d} \alpha_i \xi_i$$

for some $\alpha_i \geq 0$ with $\sum_{i=1}^d \alpha_i = 1$ and at least two of the α_i are non-zero. Hence, for any $x \in \overline{B}$,

$$x - \xi_{d+1} = \sum_{i=1}^{d} \alpha_i (x - \xi_i). \tag{8.1}$$

Regarding vectors as columns, using elementary column operations, we have the determinant relations

$$\begin{aligned}(\det M(x)) \prod_{i=1}^{d+1} |x-\xi_i| &= \begin{vmatrix} |x-\xi_1| & \cdots & |x-\xi_d| & |x-\xi_{d+1}| \\ x-\xi_1 & \cdots & x-\xi_d & x-\xi_{d+1} \end{vmatrix} \\ &= \begin{vmatrix} |x-\xi_1| & \cdots & |x-\xi_d| & \beta \\ x-\xi_1 & \cdots & x-\xi_d & 0 \end{vmatrix} \\ &= \beta \begin{vmatrix} x-\xi_1 & \cdots & x-\xi_d \end{vmatrix}.\end{aligned}$$

where $\beta = |x - \xi_{d+1}| - \sum_{i=1}^d \alpha_i |x - \xi_i|$.

For $x \in \overline{D}$, the vectors $x - \xi_1, \cdots, x - \xi_d$ are linearly independent because, if $\sum_{i=1}^d \lambda_i (x - \xi_i) = 0$, then

$$\left(\sum_{i=1}^{d} \lambda_i \right) x = \sum_{i=1}^{d} \lambda_i \xi_i.$$

If $\sum_{i=1}^d \lambda_i = 0$ then $\sum_{i=1}^d \lambda_i \xi_i = 0$ which forces $\lambda_i = 0$ from the linear independence of $\xi^1, \cdots, \xi^d$. If $\sum_{i=1}^d \lambda_i \neq 0$ then

$$x = \sum_{i=1}^{d} \sigma_i \xi_i$$

with $\sum_{i=1}^d \sigma_i = 1$ where $\sigma_i = \lambda_i / \sum_{i=1}^d \lambda_i$. This violates the hypothesis that $\overline{D}$ does not intersect $\operatorname{hyp}(\xi_1, \cdots, \xi_d)$. Hence, for $x \in \overline{D}$, the determinant $\begin{vmatrix} x-\xi_1 & \cdots & x-\xi_d \end{vmatrix}$ is non-zero.

Next, for $x \in \overline{D}$, from (8.1) and the triangle inequality, we have

$$|x - \xi_{d+1}| = \left| \sum_{i=1}^{d} \alpha_i (x - \xi_i) \right| < \sum_{i=1}^{d} \alpha_i |x - \xi_i|$$

because $\alpha_i \geq 0$, the $x - \xi_i$, $i = 1, \cdots, d$ are not parallel (because they are linearly independent as shown above) and at least two of the $\alpha_i(x - \xi_i)$ are non-zero. Hence $\beta \neq 0$.

So combining the conclusions of the previous two paragraphs, we have $\det M(x) \neq 0$ for all $x \in \overline{D}$, which completes the proof of (a).

Proof of (b).

We start with the claim that every $x \in \mathbb{R}^d$ has a unique representation as $x = \sum_{i=1}^{d+1} \alpha_i \xi_i$ for some $\alpha_i \in \mathbb{R}$ with $\sum_{i=1}^{d+1} \alpha_i = 1$. We postpone the proof of this claim to the end of this section and continue with the proof of (b).

For $x \in \overline{D}$, the invertibility of $M(x)$ is equivalent to the linear independence of the vectors

$$[|x-\xi_1|, x-\xi_1], \cdots, [|x-\xi_{d+1}|, x-\xi_{d+1}]$$

in $\mathbb{R}^{d+1}$. If there are $\lambda_1, \cdots, \lambda_{d+1} \in \mathbb{R}$ such that

$$\sum_{i=1}^{d+1} \lambda_i [|x-\xi_i|, x-\xi_i] = 0$$

then

$$\sum_{i=1}^{d+1} \lambda_i |x-\xi_i| = 0, \qquad \sum_{i=1}^{d+1} \lambda_i (x-\xi_i) = 0. \tag{8.2}$$

If $\sum_{i=1}^{d+1} \lambda_i \neq 0$, define

$$\mu_i = \frac{\lambda_i}{\sum_{i=1}^{d+1} \lambda_i}, \qquad i = 1, \cdots, d+1.$$

Then $\sum_{i=1}^{d+1} \mu_i = 1$ and (8.2) implies

$$x = \sum_{i=1}^{d+1} \mu_i \xi_i, \qquad \sum_{i=1}^{d+1} \mu_i |x-\xi_i| = 0. \tag{8.3}$$

Now the μ_i are uniquely determined because of the claim in the second paragraph of the proof of (b). Further, since $x \in \operatorname{conv}(\xi_1, \cdots, \xi_{d+1})$, we have $\mu_i \geq 0$, so the relation $\sum_{i=0}^{n} \mu_i = 1$ implies least one of the μ_i is positive. Hence the second equation in (8.3) implies that $x = \xi_i$ for at least one of the i, which contradicts our assumption that any $x \in \overline{D}$ is in the interior of $\operatorname{conv}(\xi_1, \cdots, \xi_{d+1})$.

So we must have $\sum_{i=1}^{d+1} \lambda_i = 0$; then (8.2) implies $\sum_{i=1}^{d+1} \lambda_i \xi_i = 0$. From our claim, every $x \in \mathbb{R}^d$ has a unique representation

$$x = \sum_{i=1}^{d+1} \alpha_i \xi_i$$

for some α_i with $\sum_{i=1}^{d+1} \alpha_i = 1$. However, we also have

$$x = \sum_{i=1}^{d+1} (\alpha_i + \lambda_i) \xi_i$$

with $\sum_{i=0}^{d+1} (\alpha_i + \lambda_i) = 1$. So the unique representation property implies $\lambda_i = 0$, $i = 1, \cdots, d+1$, proving (b).

It remains to prove the unique representation claim stated at the beginning of the proof of (b). We observe that if $\alpha_i \in \mathbb{R}$ with $\sum_{i=1}^{d+1} \alpha_i = 1$ then

$$\begin{aligned}\sum_{i=1}^{d+1} \alpha_i \xi_i &= \sum_{i=1}^{d} \alpha_i (\xi_i - \xi_{d+1}) + \left(\sum_{i=1}^{d+1} \alpha_i \right) \xi_{d+1} \\ &= \xi_{d+1} + \sum_{i=1}^{d} \alpha_i (\xi_i - \xi_{d+1}). \end{aligned} \tag{8.4}$$

Hence

$$\operatorname{hyp}(\xi_1, \cdots, \xi_{d+1}) = \xi_{d+1} + \operatorname{span}(\xi_1 - \xi_{d+1}, \cdots, \xi_d - \xi_{d+1}).$$

Now D is an open subset of $\mathbb{R}^d$ contained in $\operatorname{conv}(\xi_1, \cdots, \xi_{d+1})$ which is a subset of $\operatorname{hyp}(\xi_1, \cdots, \xi_{d+1})$. Hence $\xi_1 - \xi_{d+1}, \cdots, \xi_d - \xi_{d+1}$ must be a basis for $\mathbb{R}^d$. So $\operatorname{span}(\xi_1 - \xi_{d+1}, \cdots, \xi_d - \xi_{d+1}) = \mathbb{R}^d$ and $\operatorname{hyp}(\xi_1, \cdots, \xi_{d+1}) = \mathbb{R}^d$. Finally, the representation is unique because of (8.4) and the linear independence of $\xi_1 - \xi_{d+1}, \cdots, \xi_d - \xi_{d+1}$. □

9. Appendix

In this section we prove the existence of a unique distributional solution of an IVP for a second order hyperbolic PDE, along with an estimate of the solution in terms of the coefficients. This is a standard result but a statement and a proof of the result, suitable for our use, is difficult to find. We give a standard proof based on the well-posedness result for an IBVP for second order hyperbolic PDEs in [3].

We use the notation for time dependent Sobolev spaces in section 5.9.2 of [3]. Suppose $T > 0$, D is a bounded region in $\mathbb{R}^n$ with a smooth boundary, $a(x,t), q(x,t)$ are compactly supported smooth functions on $\mathbb{R}^n \times \mathbb{R}$ and $b(x,t)$ is a compactly supported smooth n-dimensional vector field on $\mathbb{R}^n \times \mathbb{R}$. Define

$$\mathcal{L} := \partial_t^2 - \Delta - a\partial_t + b \cdot \nabla + q, \qquad D_T = D \times (0,T),$$

the L^2 inner product

$$(v, w) = \int_D v(x)\, w(x)\, dx, \qquad v, w \in L^2(D),$$

and the bilinear forms

$$\begin{aligned} A[v,w;t] &= -\int_D a(x,t)\, v(x)\, w(x)\, dx, \\ B[v,w;t] &= \int_D \nabla v(x) \cdot \nabla w(x) + b(x,t) \cdot \nabla v(x)\, w(x) + q(x,t)\, v(x)\, w(x)\, dx, \end{aligned}$$

for $v, w \in H^1(D)$, $0 \leq t \leq T$. For functions $u(x,t)$ on $D \times (0,T)$, the expression $u(t)$ will denote the function $u(t) : D \to \mathbb{R}$ with $u(t)(x) = u(x,t)$.

For a function $F \in L^2(D_T)$, consider the IBVP

$$\mathcal{L}u = F, \qquad \text{on } D \times (0,T), \tag{9.1}$$

$$u(\cdot, t{=}0) = 0, \qquad u_t(\cdot, t{=}0) = 0, \qquad \text{on } D, \tag{9.2}$$

$$u = 0 \qquad \text{on } \partial D \times [0,T]. \tag{9.3}$$

A function $u \in L^2(0,T; H_0^1(D))$ with $u_t \in L^2(0,T; L^2(D))$ and $u_{tt} \in L^2(0,T; H^{-1}(D))$ is said to be a weak solution of the IBVP (9.1) - (9.3) if the following holds:

$$(i)\ (u_{tt}(t), v) + A[u_t(t), v; t] + B[u(t), v; t] = (F(t), v),\ \forall v \in H_0^1(D),\ \text{a.e. } 0 \leq t \leq T, \tag{9.4}$$

$$(ii)\ u(\cdot, 0) = 0,\ u_t(\cdot, 0) = 0. \tag{9.5}$$

Note that by Sobolev space theory, $u \in C([0,T], L^2(D))$ and $u_t \in C([0,T], H^{-1}(D))$, so (ii) makes sense. We also observe that if the weak solution u is in $H^2(D \times (0,T))$ then a standard argument shows that u satisfies (9.1) and (9.3) as functions.

Theorems 3,4,5 in Section 7.2 in [3] give a well-posedness result for this IBVP. Further, Theorem 6 in Section 7.2 in [3] gives higher order regularity if F has higher order regularity and satisfies a matching condition on $\partial D \times [0,T]$. We need only a special case of the general results in [3].

Proposition 9.1. *If $F \in L^2(D_T)$ then the IBVP (9.1) - (9.3) has a unique weak solution u. Further $u \in L^\infty(0,T; H_0^1(D))$ and $u_t \in L^\infty(0,T; L^2(U))$ with*

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \Big(\|u(t)\|_{H_0^1(D)} + \|u_t(t)\|_{L^2(D)} \Big) \leq C \|F\|_{L^2(D_T)} \tag{9.6}$$

and C determined by T and $\|[a,b,q]\|_{L^\infty(D_T)}$. Further, if m is a positive integer and

$$\partial_t^k F \in L^2(0,T; H^{m-k}(D)) \qquad \text{for } k = 0, \cdots, m, \tag{9.7}$$

$$(\partial_t^k F)(\cdot, 0)|_{\partial D} = 0 \qquad \text{for } k = 0, \cdots, m-2, \tag{9.8}$$

then $\partial_t^k u \in L^\infty(0,T; H^{m+1-k}(D))$ for $k = 0, 1, \cdots, m+1$ and we have the estimate

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \sum_{k=0}^{m+1} \|\partial_t^k u(\cdot, t)\|_{H^{m+1-k}(D)} \leq C \sum_{k=0}^{m} \|\partial_t^k F\|_{L^2(0,T; H^{m-k}(D))}, \tag{9.9}$$

with C determined by T and $\|[a,b,q]\|_{C^m(D_T)}$.

The theorems in [3] are for a general second order hyperbolic operator similar to our $\mathcal{L}$ except with Δ replaced by a general second order elliptic operator, and without the $a\partial_t$ term. However, with minor modifications (particularly to the proof of Theorem 4 in section 7.2 of [3]), the same proof works for our $\mathcal{L}$. In our proposition, we have also added the dependence of C on the coefficients, which follows easily if, in the proof, we track the dependence of the constants on the coefficients.

From Proposition 9.1 we derive the following existence result for an IVP, needed below.

Proposition 9.2. *Suppose m is a positive integer, $\partial_t^k F \in L^2(0,T;H^{m-k}(\mathbb{R}^n))$ for $k = 0,\cdots,m$ and F is compactly supported. Then the IVP*

$$\mathcal{L}u = F, \qquad \text{on } \mathbb{R}^n \times (0,T), \tag{9.10}$$

$$u(\cdot,t{=}0) = 0,\ u_t(\cdot,t{=}0) \qquad \text{on } \mathbb{R}^n \tag{9.11}$$

has a solution u with $\partial_t^k u \in L^\infty(0,T;H^{m+1-k}(\mathbb{R}^n))$ for $k = 0,1,\cdots,m+1$. Further

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \sum_{k=0}^{m+1} \|\partial_t^k u(\cdot,t)\|_{H^{m+1-k}(\mathbb{R}^n)} \leq C \sum_{k=0}^{m} \|\partial_t^k F\|_{L^2(0,T;H^{m-k}(\mathbb{R}^n))}, \tag{9.12}$$

with C determined by T and $\|[a,b,q]\|_{C^m(\mathbb{R}^n\times[0,T])}$.

Proof. Suppose F is supported in $B_R \times [0,T]$ where B_R is the origin centered ball of radius R. Let D be the origin centered ball of radius $2R+T$. Then F satisfies the conditions of Proposition 9.1 so the IBVP (9.1) - (9.3) has a solution u with $\partial_t^k u \in L^\infty(0,T;H^{m+1-k}(D))$ for $k = 0,1,\cdots,m+1$ and

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \sum_{k=0}^{m+1} \|\partial_t^k u(\cdot,t)\|_{H^{m+1-k}(D)} \leq C \sum_{k=0}^{m} \|\partial_t^k F\|_{L^2(0,T;H^{m-k}(D))}, \tag{9.13}$$

with C determined by T and $\|[a,b,q]\|_{C^m(D_T)}$.

Since $m \geq 1$, we have $u \in H^2(D_T)$, so $u_t^2 + |\nabla u|^2$ and $u_t \nabla u$ are in $W^{1,1}(D_T)$. Since the divergence theorem is valid, on regions with Lipschitz boundary, for vector fields with components in $W^{1,1}(D_T)$, using a standard energy estimate argument on a truncated cone and that $F = 0$ outside $B_R \times [0,T]$, one can show that $u(x,t) = 0$ for $R+T \leq |x| \leq 2R+T$, $0 \leq t \leq T$. Hence if we define $u = 0$ for $|x| \geq 2R+T$, $0 \leq t \leq T$, then we have a solution of the IVP (9.10), (9.11) with the properties claimed in Proposition 9.2. ☐

Suppose F is a distribution on $\mathbb{R}^n \times (-\infty,T)$ with $F = 0$ for $t < 0$. Consider the IVP

$$\mathcal{L}u = F, \qquad \text{on } \mathbb{R}^n \times (-\infty,T), \tag{9.14}$$

$$u = 0 \qquad \text{on } \mathbb{R}^n \times (-\infty,0). \tag{9.15}$$

We say a distribution u on $\mathbb{R}^n \times (-\infty,T)$ is a solution of this IVP if $u = 0$ for $t < 0$ and

$$\langle u, \mathcal{L}^*\phi\rangle = \langle F,\phi\rangle, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^n \times (-\infty,T));$$

here $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$. We have the following well-posedness result for the IVP (9.14), (9.15).

Proposition 9.3. *Suppose m is a positive integer, $\partial_t^k F \in L^2(-\infty,T;H^{m-k}(\mathbb{R}^n))$ for $k = 0,\cdots,m$, F compactly supported and $F = 0$ for $t < 0$. Then the IVP* (9.14), (9.15) *has a unique distributional solution u. Further, if $m > (n-1)/2$ then for any non-negative integer $p < m-(n-1)/2$ we have $u \in C^p(\mathbb{R}^n \times (-\infty,T])$ and*

$$\|u\|_{C^p(\mathbb{R}^n\times(-\infty,T])} \leq C \sum_{k=0}^{m} \|\partial_t^k F\|_{L^2(0,T;H^{m-k}(\mathbb{R}^n))},$$

with C determined by T and $\|[a,b,q]\|_{C^m(\mathbb{R}^n\times[0,T])}$.

Proof. If we apply Proposition 9.2 with the initial condition $u(\cdot,-\epsilon)=0,\ u_t(\cdot,-\epsilon)=0$ for some $\epsilon>0$, then we are guaranteed a solution $u\in H^{m+1}(\mathbb{R}^n\times(-\epsilon,T))$ with

$$\|u\|_{H^{m+1}(\mathbb{R}^n\times(-\epsilon,T))}\leq C\sum_{k=0}^{m}\|\partial_t^k F\|_{L^2(-\epsilon,T;H^{m-k}(\mathbb{R}^n))},$$

with C determined by T and $\|[a,b,q]\|_{C^m(\mathbb{R}^n\times[-\epsilon,T])}$. Since $m\geq 1$, we have $u\in H^2$, so by an energy estimate u will be zero for $-\epsilon<t<0$. We extend u as the zero function for the region $t\leq-\epsilon$; then u is a distributional solution of (9.14), (9.15).

Suppose $m>(n-1)/2$. Noting that u is compactly supported with the support determined by T and the support of F, from the Sobolev embedding theorem, for any non-negative integer $p<m+1-(n+1)/2=m-(n-1)/2$, we have $u\in C^p(\mathbb{R}^n\times(-\infty,T])$ and

$$\|u\|_{C^p(\mathbb{R}^n\times(-\infty,T])}\leq C\sum_{k=0}^{m}\|\partial_t^k F\|_{L^2(0,T;H^{m-k}(\mathbb{R}^n))},$$

with C determined by T and $\|[a,b,q]\|_{C^m(\mathbb{R}^n\times[0,T])}$. It remains to prove the uniqueness of the distributional solution.

Note that for F regular enough, there is a C^2 solution of (9.14), (9.15). Further, a standard energy estimate shows that there is at most one C^2 solution. This will be important for us in our proof next of the claim that if $F=0$ then any distributional solution u of (9.14), (9.15) must be zero.

Suppose ϕ is a compactly supported smooth function on $\mathbb{R}^n\times\mathbb{R}$ with support $0\leq t\leq T$. Consider the backward IVP

$$\begin{aligned}\mathcal{L}^*v&=\phi \qquad &&\mathbb{R}^n\times(-3,\infty)\\ v&=0 &&\text{on } t>T.\end{aligned}$$

Then, reversing time t and using the existence part (for arbitrary large m) and the uniqueness of C^2 solutions (proved already), we know there is a smooth solution v on $\mathbb{R}^n\times(-3,\infty)$ of this backward IVP. Further, the restriction of v to $t\geq t_1$ is compactly supported for any $t_1>-3$.

Let $\rho(t)$ be a smooth function on $\mathbb{R}$ with

$$\rho(t)=\begin{cases}1, & t>-1\\ 0, & t<-2\end{cases},$$

and define $w(x,t)=\rho(t)v(x,t)$. Then w is a compactly supported smooth function on $\mathbb{R}^n\times\mathbb{R}$ and, on the region $t>-1$ we have $\mathcal{L}^*(w)=\mathcal{L}^*v=\phi$. Noting that $u=0$ for $t<0$ and using the definition of a distributional solution, we have

$$\langle u,\phi\rangle=\langle u,\mathcal{L}^*w\rangle=0, \qquad \forall\phi\in C_c^\infty(\mathbb{R}^n\times(-\infty,T)).$$

Hence $u = 0$ on $\mathbb{R}^n \times (-\infty, T)$. □

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† TIFR Centre for Applicable Mathematics, Bangalore 560065, India.
E-mail: vkrishnan@tifrbng.res.in

* Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA.
E-mail: rakesh@udel.edu

‡ TIFR Centre for Applicable Mathematics, Bangalore 560065, India.
E-mail: soumen@tifrbng.res.in