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Determination of Lower Order Perturbations of the Polyharmonic Operator from Partial Boundary Data

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We consider a perturbed polyharmonic operator $\mathcal{L}(x,D)$ of order $2m$ defined on a bounded simply connected domain $\Omega \subset \mathbb{R}^n, n \geq 3$ with smooth connected boundary of the form:

$$\mathcal{L}(x,D) = (-\Delta)^m + \sum_{k=1}^{[\![\frac{m}{2}]\!]} b_k(x)(-\Delta)^k + \sum_{k=1}^{[\![\frac{m+1}{2}]\!]} (A_k(x)\cdot D)(-\Delta)^{k-1} + q(x),$$

where $D = -\mathrm{i}\nabla$ and $[\![\cdot]\!]$ stands for the greatest integer function. In the biharmonic case, such operators arise in the study of certain elasticity and buckling problems. We study an inverse problem involving $\mathcal{L}$ and show that all the coefficients b_k, A_k and q can be recovered from partial Dirichlet-to-Neumann (D-N) data on the boundary.

Keywords: Calderón-type inverse problem; Perturbed polyharmonic operator; Carleman estimates

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1. Introduction and statements of the main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded simply connected domain with smooth connected boundary. Let us consider the following perturbed polyharmonic operator $\mathcal{L}(x,D)$ of order $2m$ with m^{th} order perturbations of the form:

$$\mathcal{L}(x,D) = (-\Delta)^m + \sum_{k=1}^{[\![\frac{m}{2}]\!]} b_k(x)(-\Delta)^k + \sum_{k=1}^{[\![\frac{m+1}{2}]\!]} (A_k(x)\cdot D)(-\Delta)^{k-1} + q(x), \tag{1.1}$$

where $D = -\mathrm{i}\nabla$ and the coefficients, $b_k \in C^{2m+(2k-1)(m+1)}(\overline{\Omega},\mathbb{C})$, $A_k \in C^{2m+(2k-2)(m+1)}(\overline{\Omega},\mathbb{C}^n)$ and $q \in L^\infty(\Omega,\mathbb{C})$.

The operator $\mathcal{L}(x,D)$ with the domain $\mathcal{D}(\mathcal{L}(x,D))$, where,

$$\mathcal{D}(\mathcal{L}(x,D)) := \Big\{ u \in H^{2m}(\Omega) \mid u|_{\partial\Omega} = -\Delta u|_{\partial\Omega} = \cdots = (-\Delta)^{m-1}u|_{\partial\Omega} = 0 \Big\}$$

is an unbounded closed operator on $L^2(\Omega)$ with a purely discrete spectrum [6]. We make the assumption that 0 is not an eigenvalue of the operator $\mathcal{L}(x,D) : \mathcal{D}(\mathcal{L}(x,D)) \to L^2(\Omega)$.

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Then for any $f = (f_0, f_1, .., f_{m-1}) \in \prod_{i=0}^{m-1} H^{2m-2i-\frac{1}{2}}(\partial\Omega)$, the boundary value problem

$$\begin{aligned} &\mathcal{L}(x,D)u = 0 \text{ in } \Omega \\ &u|_{\partial\Omega} = f_0, (-\Delta)u|_{\partial\Omega} = f_1, \cdots, (-\Delta)^{m-1}u|_{\partial\Omega} = f_{m-1} \end{aligned} \tag{1.2}$$

has a unique solution $u \in H^{2m}(\Omega)$.

Let us define the corresponding Neumann trace operator $\gamma^{\#}$ by

$$\gamma^{\#}u = \Big(\partial_\nu u|_{\partial\Omega}, \cdots, \partial_\nu(-\Delta)^k u|_{\partial\Omega}, \cdots, \partial_\nu(-\Delta)^{m-1}u|_{\partial\Omega}\Big)$$

where ν is the outer unit normal to the boundary $\partial\Omega$, and the corresponding Dirichlet-to-Neumann (D-N) map by

$$\begin{aligned} &\mathcal{N} : \prod_{i=0}^{m-1} H^{2m-2i-\frac{1}{2}}(\partial\Omega) \to \prod_{i=0}^{m-1} H^{2m-2i-\frac{3}{2}}(\partial\Omega) \\ &\mathcal{N}(f) = \gamma^{\#}u = \Big(\partial_\nu u|_{\partial\Omega}, \cdots, \partial_\nu(-\Delta)^k u|_{\partial\Omega}, \cdots, \partial_\nu(-\Delta)^{m-1}u|_{\partial\Omega}\Big). \end{aligned} \tag{1.3}$$

We are interested in the unique recovery of the coefficients in (1.1) from the D-N map given on a part of the boundary $\partial\Omega$. Recovery of first order perturbations of the biharmonic and polyharmonic operators in dimensions $n \geq 3$ has been considered in prior works [7, 8, 10, 11, 15]. In [7, 8], recovery of the zeroth order perturbations of the biharmonic operator was studied and recently in [10, 11], recovery of first order perturbations of the biharmonic operator with partial D-N data and of the polyharmonic operator with full D-N data were considered. Recovery of first order perturbations of the biharmonic operator from partial D-N data was shown in an infinite slab and for certain bounded domains in [15]. A natural question to ask is whether higher order perturbations of the polyharmonic operator can be recovered from partial D-N data. In this paper, we show that all the coefficients of (1.1) can be recovered from partial D-N data. To the best of the authors' knowledge, inverse problems involving higher order perturbations (order ≥ 2) of the polyharmonic operator have not been investigated in previous studies. We consider the results of this paper as a natural generalization of the results obtained in [10, 11] which considered the recovery of only zeroth and first order perturbations. It is an interesting open question whether other forms of higher order perturbations of the polyharmonic operator can be recovered from full or partial D-N data. The authors are of the opinion that some new techniques are required to deal with such higher order perturbations and the methods used in this paper alone are not sufficient.

In the biharmonic case, equations of the kind considered here (1.1) come up in the study of continuum mechanics of elasticity and buckling problems [1, 5, 14]. Inverse problems with partial boundary information arise naturally in several imaging applications including seismic and medical imaging, electrical impedance tomography to name a few. The techniques we rely on to prove our main result are based on the pioneering works done for inverse boundary value problems involving Schrödinger operators [2–4, 9, 13], more specifically, interior and boundary Carleman estimates, and complex geometric optics (CGO) solutions. We extend the boundary Carleman estimates proved for the biharmonic case [10] to the perturbed polyharmonic operator with m lower order perturbations as in (1.1). Even in the biharmonic case, the boundary Carleman estimate here includes the result in [10] as we consider a second order Laplacian perturbation. To deal with the recovery of m lower order terms given the boundary data (1.3) when all its components are measured on the partial boundary, our ansatz for the CGO solution of (1.1) has m lower order terms; see (2.9).

We now state the main result of this article. Following [9], we define the front face $F(x_0)$ of the boundary $\partial\Omega$ with respect to a point $x_0 \in \mathbb{R}^n \setminus \overline{\mathrm{ch}(\Omega)}$ where $\mathrm{ch}(\Omega)$ is the convex hull of Ω. We let

$$F(x_0) = \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) \leq 0\}, \tag{1.4}$$

where $\nu(x)$ is the unit outer normal to $\partial\Omega$.

Let $F^{\#}$ be an open neighborhood of $F(x_0)$ in $\partial\Omega$. Our main result is:

THEOREM 1.1 *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded simply connected domain with smooth connected boundary. Let $\mathcal{L}(x, D)$ and $\widetilde{\mathcal{L}}(x, D)$ be two operators defined as in (1.1) with the coefficients $b_k, \widetilde{b}_k \in C^{2m+(2k-1)(m+1)}(\overline{\Omega}, \mathbb{C})$ and $A_k, \widetilde{A}_k \in C^{2m+(2k-2)(m+1)}(\overline{\Omega}, \mathbb{C}^n)$ and $q, \widetilde{q} \in L^\infty(\Omega, \mathbb{C})$ respectively. We assume that 0 is not an eigenvalue of the operators, $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to L^2(\Omega)$ and $\widetilde{\mathcal{L}} : \mathcal{D}(\widetilde{\mathcal{L}}) \to L^2(\Omega)$. Let $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ be the corresponding D-N maps of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, respectively, satisfying*

$$\mathcal{N}(f)|_{F^{\#}} = \widetilde{\mathcal{N}}(f)|_{F^{\#}} \quad \text{for } f \in \prod_{i=0}^{m-1} H^{2m-2i-\frac{1}{2}}(\partial\Omega)$$

where

$$\mathcal{N}(f)|_{F^{\#}} = \Big(\partial_\nu u|_{F^{\#}}, \cdots, \partial_\nu(-\Delta)^l u|_{F^{\#}}, \cdots, \partial_\nu(-\Delta)^{m-1} u|_{F^{\#}}\Big). \tag{1.5}$$

Then for each k

$$b_k = \widetilde{b}_k, \quad A_k = \widetilde{A}_k \quad \text{and} \quad q = \widetilde{q} \quad \text{in} \quad \Omega.$$

This paper is organized as follows. Section 2 is devoted to the proof of interior Carleman estimates which is used in the proof of existence of complex geometric optics (CGO) solutions for (1.2). We end Section 2 by deriving boundary Carleman estimates required to deal with partial boundary Neumann data. In Section 3, we derive an integral identity involving the coefficients to be determined and then give the proof of the uniqueness result, Theorem 1.1.

We end this section with two corollaries of Theorem 1.1. These are similar to the results in [10] and follow from arguments given in [10] once Theorem 1.1 is proven. We will not repeat the arguments here.

COROLLARY 1.2 *Consider Ω as above and let $x_1 \in \partial\Omega$ be a point such that the tangent plane H of $\partial\Omega$ at x_1 satisfies $\partial\Omega \cap H = \{x_1\}$. Also assume that Ω is strongly star-shaped with respect to x_1 (that is, every line through x_1 which is not contained in the tangent plane H cuts the boundary $\partial\Omega$ at precisely two distinct points, x_1 and x_2, and the intersection at x_2 is transversal). With the same assumptions on the coefficients as in Theorem 1.1 and assuming that there exists a neighborhood $\widetilde{F} \subset \partial\Omega$ of x_1 such that*

$$\mathcal{N}(f)|_{\widetilde{F}} = \widetilde{\mathcal{N}}(f)|_{\widetilde{F}} \text{ for all } f \in \prod_{i=0}^{2m-1} H^{2m-2i-\frac{1}{2}}(\partial\Omega).$$

Then we have

$$b_k = \widetilde{b}_k, \quad A_k = \widetilde{A}_k, \quad \text{and} \quad q = \widetilde{q} \quad \text{in} \quad \Omega.$$

Finally we consider a boundary value problem for $\mathcal{L}$ but with Dirichlet boundary conditions:

$$\begin{aligned} &\mathcal{L}(x, D)u = 0 \text{ in } \Omega \\ &\gamma_D u = (u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}, .., \partial_\nu^{m-1} u|_{\partial\Omega}) = (f_0, f_1, .., f_{m-1}) \in \prod_{i=0}^{m-1} H^{2m-i-\frac{1}{2}}(\partial\Omega). \end{aligned} \tag{1.6}$$

The corresponding Neumann trace is

$$\gamma_D^\# = \{\partial_\nu^m u|_{\partial\Omega}, .., \partial_\nu^{2m-1} u|_{\partial\Omega}\} \in \prod_{i=m}^{2m-1} H^{2m-i-\frac{1}{2}}(\partial\Omega).$$

Introduce the set of Cauchy data C^D for the operator $\mathcal{L}(x, D)$ with Dirichlet boundary conditions by

$$C^D = \left\{(u|_{\partial\Omega}, \cdots, \partial_\nu^{m-1} u|_{\partial\Omega}, \partial_\nu^m u|_{\partial\Omega}, \cdots, \partial_\nu^{2m-2} u|_{\partial\Omega}, \partial_\nu^{2m-1} u|_{F^\#})\right\}$$

where $u \in H^{2m}(\Omega)$ solves (1.6) and $F^\#$ is an open neighborhood of $F(x_0)$ defined in (1.4). We make the assumption that 0 is not an eigenvalue of $\mathcal{L}$ from $\mathcal{D}_D(\mathcal{L}) \to L^2(\Omega)$, where

$$\mathcal{D}_D(\mathcal{L}) = \{u \in H^{2m}(\Omega) : \gamma_D u = 0\}.$$

We have the following result.

Corollary 1.3 *Assume that for each k, $b_k, \widetilde{b}_k$, $A_k, \widetilde{A}_k$ and $q, \widetilde{q}$ satisfy the conditions of Theorem 1.1 and that 0 is not an eigenvalue of the operators, $\mathcal{L} : \mathcal{D}_D(\mathcal{L}) \to L^2(\Omega)$ and $\widetilde{\mathcal{L}} : \mathcal{D}_D(\widetilde{\mathcal{L}}) \to L^2(\Omega)$. Then $C^D = \widetilde{C}^D$ implies that for each k, $b_k = \widetilde{b}_k$, $A_k = \widetilde{A}_k$ and $q = \widetilde{q}$ in Ω.*

2. Carleman estimates and CGO solutions

This section is devoted to deriving Complex Geometric Optics (CGO) solutions for the operator $\mathcal{L}$ as well as for its formal L^2 adjoint $\mathcal{L}^*$ based on Carleman estimates. In the latter part of this section, we derive boundary Carleman estimates to deal with partial boundary data.

2.1. *Interior Carleman estimates*

As is easily seen, $\mathcal{L}^*$ does not have the same form as that of $\mathcal{L}$. To deal with both $\mathcal{L}$ and $\mathcal{L}^*$ in a unified manner, we consider the following operator

$$\mathcal{L}_\#(x, D) = (-\Delta)^m + \sum_{|l| \le m} c_l(x) D^l \tag{2.1}$$

where $l = (l_1, .., l_n)$ is a multi-index and $c_l \in C^{2m+(|l|-1)(m+1)}(\overline{\Omega}, \mathbb{C})$, and derive interior Carleman estimates for the semiclassical version of this operator.

First we will focus on the semiclassical version of the principal part of this perturbed operator, namely, $(-h^2\Delta)^m$ and then will add lower order terms to it to get the Carleman estimate for the operator $\mathcal{L}_{\#}(x, hD)$.

We start by recalling the definition of a limiting Carleman weight for the semiclassical Laplacian. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain with smooth boundary. Let $\widetilde{\Omega}$ be an another open set in $\mathbb{R}^n$ such that $\Omega \subset\subset \widetilde{\Omega}$ and $\varphi \in C^\infty(\widetilde{\Omega}, \mathbb{R})$, and consider the conjugated operator

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}} \tag{2.2}$$

with its semiclassical symbol $p_\varphi(x, \xi)$.

Definition 1 [9] We say that φ is a limiting Carleman weight for $-h^2\Delta$ in $\widetilde{\Omega}$ if $\nabla\varphi \neq 0$ in $\widetilde{\Omega}$ and the Poisson bracket of $\mathrm{Re}(p_\varphi)$ and $\mathrm{Im}(p_\varphi)$ satisfies

$$\Big\{\mathrm{Re}(p_\varphi), \mathrm{Im}(p_\varphi)\Big\}(x, \xi) = 0 \text{ when } p_\varphi(x, \xi) = 0 \text{ for } (x, \xi) \in (\overline{\Omega} \times \mathbb{R}^n).$$

We use the semiclassical Sobolev spaces $H^s(\mathbb{R}^n)$ with $s \in \mathbb{R}^n$ equipped with the norm $\|u\|_{H^s(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2}$ where $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

We now prove the following proposition.

Proposition 2.1 *Suppose c_l for $1 \leq |l| \leq m$ and c_0 in (2.1) satisfy $c_l \in C^{2m+(|l|-1)(m+1)}(\overline{\Omega}, \mathbb{C})$ and $c_0 \in L^\infty(\Omega, \mathbb{C})$. Let φ be a limiting Carleman weight for the semiclassical Laplacian on $\widetilde{\Omega}$. Then for $0 < h \ll 1$ and $-2m \leq s \leq 0$*

$$h^m\|u\|_{H^{s+2m}_{\mathrm{scl}}} \leq C\|h^{2m}e^{\frac{\varphi}{h}}\mathcal{L}_{\#}(x, D)e^{-\frac{\varphi}{h}}u\|_{H^s_{\mathrm{scl}}}, \text{ for } u \in C_0^\infty(\Omega), \tag{2.3}$$

where the constant $C = C_{s,\Omega,c_l}$ is independent of h.

Proof. We consider the convexified Carleman weight φ_ε considered in [9]

$$\varphi_\varepsilon = \varphi + \frac{h}{2\varepsilon}\varphi^2 \text{ on } \widetilde{\Omega}$$

and use the Carleman estimate for the semiclassical Laplacian with a gain of two derivatives proven in [12, Lemma 2.1] and iterate it m times to get the following estimate:

$$(\frac{h}{\sqrt{\epsilon}})^m||u||_{H^{s+2m}_{\mathrm{scl}}} \leq C||e^{\frac{\varphi_\varepsilon}{h}}(-h^2\Delta)^m e^{-\frac{\varphi_\varepsilon}{h}}u||_{H^s_{\mathrm{scl}}}, \text{ for } u \in C_0^\infty(\Omega) \text{ and } s \in \mathbb{R}. \tag{2.4}$$

Now let $-2m \leq s \leq 0$ and we now show that we can add lower order terms of (2.1) to get the estimate in (2.3).

We first consider addition of the zeroth order ($|l| = 0$) term $h^{2m}c_0$, where $c_0 \in L^\infty(\Omega, \mathbb{C})$. We have

$$\|h^{2m}c_0u\|_{H^s_{\mathrm{scl}}} \leq h^{2m}\|c_0\|_{L^\infty}\|u\|_{L^2} \leq h^{2m}\|c_0\|_{L^\infty}\|u\|_{H^{s+2m}_{\mathrm{scl}}}.$$

Next we consider the first order ($|l| = 1$) term $h^{2m}c_lD^l$ where $c_l \in C^{2m}(\overline{\Omega}, \mathbb{C})$.

Denoting by $A = (c_{1,0,\cdots,0}, \cdots, c_{0,\cdots,0,1}) \in C^{2m}(\overline{\Omega}, \mathbb{C}^n)$, we have

$$h^{2m} e^{\frac{\varphi_\varepsilon}{h}} \sum_{|l|=1} c_l(x) D^l e^{-\frac{\varphi_\varepsilon}{h}} = h^{2m-1} e^{\frac{\varphi_\epsilon}{h}} (A \cdot hD) e^{-\frac{\varphi_\epsilon}{h}} = h^{2m-1}(\mathrm{i}A \cdot \nabla\varphi_\epsilon + A \cdot hD). \quad (2.5)$$

Note that $\|A \cdot \nabla\varphi_\epsilon\|_{L^\infty} = \mathcal{O}(1)$ since $\nabla\varphi_\epsilon = \left(1 + \frac{h}{\epsilon}\varphi\right)\nabla\varphi$, $\varphi \in C^\infty(\overline{\Omega})$ and $0 < h \ll \epsilon \ll 1$. Hence the first term in the equation above can be estimated as follows:

$$\|(A \cdot \nabla\varphi_\epsilon)u\|_{H^s_{\mathrm{scl}}} \leq \|A \cdot \nabla\varphi_\epsilon\|_{L^\infty} \|u\|_{H^s_{\mathrm{scl}}} \leq \mathcal{O}(1)\|u\|_{H^{s+2m}_{\mathrm{scl}}} \text{ since } -2m \leq s \leq 0.$$

For the second term on the right in (2.5), we use as in [10],

$$\begin{aligned}\|A \cdot hDu\|_{H^s_{\mathrm{scl}}} &\leq \sum_{i=1}^n \|hD_i(A_i u)\|_{H^s_{\mathrm{scl}}} + h\|(\mathrm{div}A)u\|_{H^s_{\mathrm{scl}}} \\ &\leq \mathcal{O}(1) \sum_{i=1}^n \|A_i u\|_{H^{s+1}_{\mathrm{scl}}} + \mathcal{O}(h)\|u\|_{H^{s+2m}_{\mathrm{scl}}} \leq \mathcal{O}(1)\|u\|_{H^{s+2m}_{\mathrm{scl}}}\end{aligned}$$

where the last inequality follows from continuity of the multiplication operator $A_i : H^{s+2m}_{\mathrm{scl}} \to H^{s+1}_{\mathrm{scl}}$. Therefore

$$||h^{2m-1} e^{\frac{\varphi_\epsilon}{h}} (A \cdot hD) e^{-\frac{\varphi_\epsilon}{h}} u||_{H^s_{\mathrm{scl}}} \leq \mathcal{O}(h^{2m-1})||u||_{H^{s+2m}_{\mathrm{scl}}}. \quad (2.6)$$

Next we consider the second order term $h^{2m} c_l D^l$ with $c_l \in C^{3m+1}(\overline{\Omega}, \mathbb{C})$. We can view the coefficients c_l as entries of a matrix $(B_{ij})_{1 \leq i,j \leq n}$. We then have

$$\begin{aligned}e^{\frac{\varphi_\epsilon}{h}} \Big\{ \sum_{i,j=1}^n B_{ij} h^2 D_i D_j \Big\} e^{-\frac{\varphi_\epsilon}{h}} &= \Big\{ \sum_{ij} B_{ij} h^2 D_i D_j + 2\mathrm{i} \sum_{ij} B_{ij} hD_i \frac{\partial\varphi_\varepsilon}{\partial x_j} \\ &\quad - \sum_{ij} B_{ij} \frac{\partial\varphi_\varepsilon}{\partial x_i} \frac{\partial\varphi_\varepsilon}{\partial x_j} + h \sum_{ij} B_{ij} \frac{\partial^2\varphi_\varepsilon}{\partial x_i \partial x_j} \Big\}.\end{aligned}$$

For the first term above, arguing in a similar fashion as we did in the addition of the first order term, we have,

$$\begin{aligned}\|\sum_{ij} B_{ij} h^2 D_i(D_j u)\|_{H^s_{\mathrm{scl}}} &= \|\sum_{ij} \{hD_i(B_{ij} hD_j u) - hD_i(B_{ij}) hD_j u\}\|_{H^s_{\mathrm{scl}}} \\ &\leq \sum_i \|hD_i\Big(\sum_j (B_{ij} hD_j u)\Big)\|_{H^s_{\mathrm{scl}}} \\ &\quad + \mathcal{O}(h) \sum_i \|\sum_j D_i(B_{ij}) hD_j u\|_{H^s_{\mathrm{scl}}} \\ &\leq \sum_i \|\sum_j (B_{ij}) hD_j u\|_{H^{s+1}_{\mathrm{scl}}} + \mathcal{O}(h) \sum_i ||u||_{H^{s+1}_{\mathrm{scl}}} \\ &\leq \mathcal{O}(1)\|u\|_{H^{s+2m}_{\mathrm{scl}}}.\end{aligned}$$

Similarly the second term can be estimated as

$$\|\sum_{ij} B_{ij}\frac{\partial}{\partial x_i}\varphi_\epsilon hD_j u\|_{H^s_{\rm scl}} = \|\sum_j \Big(\sum_i B_{ij}\frac{\partial \varphi_\varepsilon}{\partial x_i}\Big)hD_j u\|_{H^s_{\rm scl}} \le \mathcal{O}(1)\|u\|_{H^{s+2m}_{\rm scl}}$$

and for the remaining terms, we have

$$\|\sum_{ij} B_{ij}\frac{\partial \varphi_\varepsilon}{\partial x_i}\frac{\partial \varphi_\varepsilon}{\partial x_j}u\|_{H^s_{\rm scl}} \le \|\sum_{ij} B_{ij}\frac{\partial \varphi_\varepsilon}{\partial x_i}\frac{\partial \varphi_\varepsilon}{\partial x_j}\|_{L^\infty}\|u\|_{H^{s+2m}_{\rm scl}} \le \mathcal{O}(1)\|u\|_{H^{s+2m}_{\rm scl}}$$

$$\|h\sum_{ij} B_{ij}\frac{\partial^2 \varphi_\varepsilon}{\partial x_i \partial x_j}u\|_{H^s_{\rm scl}} \le \mathcal{O}(h)\|\sum_{ij} B_{ij}\frac{\partial^2 \varphi_\varepsilon}{\partial x_i \partial x_j}\|_{L^\infty}\|u\|_{H^{s+2m}_{\rm scl}} \le \mathcal{O}(1)\|u\|_{H^{s+2m}_{\rm scl}}.$$

Therefore

$$\|h^{2m-2}e^{\frac{\varphi_\epsilon}{h}}BD\cdot De^{-\frac{\varphi_\epsilon}{h}}u\|_{H^s_{\rm scl}} \le Oc(h^{2m-2})\|u\|_{H^{s+2m}_{\rm scl}}. \tag{2.7}$$

Now in a similar way we can add all the lower order perturbation terms in a successive way, whose derivatives are at most order m to get the following:

$$\|h^{2m}e^{\frac{\varphi_\epsilon}{h}}c_l D^l e^{-\frac{\varphi_\epsilon}{h}}u\|_{H^s_{\rm scl}} \le \mathcal{O}(h^{2m-|l|})\|u\|_{H^{s+2m}_{\rm scl}}. \tag{2.8}$$

The essential idea is to use Leibniz rule as before and use the interpolation inequality. Then adding all the lower order terms up to order m in (2.4) and choosing $h \ll \epsilon \ll 1$ small enough and using the standard bounds ($1 \le e^{\frac{\varphi^2}{2\epsilon}} \le C$, $\frac{1}{2} \le 1 + \frac{h}{\epsilon}\varphi \le \frac{3}{2}$ we get the estimate in (2.3). ■

Let us denote

$$\mathcal{L}_{\#,\varphi}(x,D) = h^{2m}e^{\frac{\varphi}{h}}\mathcal{L}_{\#}(x,D)e^{-\frac{\varphi}{h}}.$$

By straightforward computation one has that the formal L^2 adjoint of $\mathcal{L}_{\#,\varphi}$ has a similar form as $\mathcal{L}_{\#,\varphi}$ with φ replaced by $-\varphi$. Since $-\varphi$ is also a limiting Carleman weight if φ is, the Carleman estimate derived in Proposition 2.1 holds for $\mathcal{L}_{\#,-\varphi}$ as well. The following proposition establishes an existence result for an inhomogeneous equation, analogous to the results in [4, 9–11].

PROPOSITION 2.2 *Suppose c_l for $1 \le |l| \le m$ and c_0 in (2.1) satisfy $c_l \in C^{2m+(|l|-1)(m+1)}(\overline{\Omega},\mathbb{C})$ and $c_0 \in L^\infty(\Omega,\mathbb{C})$. Let φ be a limiting Carleman weight for the semiclassical Laplacian on $\widetilde{\Omega}$. Then for $0 < h \ll 1$, the equation*

$$\mathcal{L}_{\#,\varphi}(x,D)u = v \ \text{ in } \Omega,$$

for $v \in L^2(\Omega)$ has a solution $u \in H^{2m}(\Omega)$ satisfying $h^m\|u\|_{H^{2m}_{\rm scl}} \le C\|v\|_{L^2(\Omega)}$.

2.2. *Construction of CGO solutions*

Next we construct complex geometric optics solutions to the equation $\mathcal{L}_{\#}(x,D)u = 0$ based on Proposition 2.2. Our solution ansatz is of the form:

$$u = e^{\frac{(\varphi + i\psi)}{h}}(a_0(x) + ha_1(x) + h^2a_2(x) + .. + h^{m-1}a_{m-1}(x) + r(x;h)). \tag{2.9}$$

Here recall that $0 < h \ll 1$, $\varphi(x)$ is a limiting Carleman weight for the semiclassical Laplacian, the real valued phase function ψ is chosen such that ψ remains smooth near $\overline{\Omega}$ and it solves the following eikonal equation $p_\varphi(x, \nabla\psi) = 0$ in $\widetilde{\Omega}$. Recall that p_φ is the semiclassical symbol of (2.2). The functions $a_0, a_1, ..a_{m-1}$ are complex amplitudes that are the solutions of m transport equations which we will define later and r is the remainder term which satisfies the estimate $\|r\|_{H^{2m}_{\mathrm{scl}}} = \mathcal{O}(h^m)$.

Following [4, 9, 10], we consider φ and ψ to be

$$\varphi(x) = \frac{1}{2}\log|x - x_0|^2, \quad \psi(x) = \mathrm{dist}_{\mathbb{S}^{n-1}}\left(\frac{x - x_0}{|x - x_0|}, \omega\right), \tag{2.10}$$

where $x_0 \in \mathbb{R}^n \setminus \overline{\mathrm{ch}(\Omega)}$ and $\omega \in \mathbb{S}^{n-1}$ is chosen such that ψ remains smooth near $\overline{\Omega}$ and it solves the eikonal equation $p_\varphi(x, \nabla\psi) = 0$ in $\widetilde{\Omega}$, that is, $|\nabla\varphi| = |\nabla\psi|$ and $\nabla\varphi \cdot \nabla\psi = 0$.

Note that the formal L^2 adjoint of $\mathcal{L}^*$ of $\mathcal{L}$ can be written in the form:

$$\mathcal{L}^* = \begin{cases} (-\Delta)^m + \overline{b_{\frac{m}{2}}}(x)(-\Delta)^{\frac{m}{2}} + \sum_{|l|\le m-1} c_l D^l, & \text{for } m \text{ even;} \\ (-\Delta)^m + (\overline{A_{\frac{m+1}{2}}(x)} \cdot D)(-\Delta)^{\frac{m-1}{2}} + \sum_{|l|\le m-1} c_l D^l, & \text{for } m \text{ odd.} \end{cases}$$

To deal with the even and odd cases simultaneously, we prove the following proposition for a slightly more general operator, which we still let as $\mathcal{L}_\#$.

Proposition 2.3 *Consider the equation*

$$\mathcal{L}_\#(x, D)u = (-\Delta)^m u + \sum_{|l|+2=m} \widetilde{c}_l D^l \circ (-\Delta)u + \sum_{|l|\le m-1} c_l D^l u = 0 \tag{2.11}$$

where the coefficients $c_l \in C^{2m+(|l|-1)(m+1)}(\overline{\Omega}, \mathbb{C})$ *with* $1 \le |l| \le m-1$, $\widetilde{c}_l \in C^{2m+(m-1)(m+1)}(\overline{\Omega}, \mathbb{C})$ *and* $c_0 \in L^\infty(\Omega, \mathbb{C})$. *Then for all* $0 < h \ll 1$, *there exists a solution* u *of* (2.11) *of the form*

$$u(x,h) = e^{\frac{(\varphi(x)+i\psi(x))}{h}}(a_0(x) + ha_1(x) + \cdots + h^{m-1}a_{m-1}(x) + r(x;h)) \in H^{2m}(\Omega)$$

where φ *and* ψ *are as in* (2.10) *and the functions* $a_0, \cdots, a_{m-1}$ *are complex amplitudes satisfying certain transport equations and* r *is a remainder term satisfying the estimate* $\|r\|_{H^{2m}_{\mathrm{scl}}} = \mathcal{O}(h^m)$.

Proof. We consider the conjugated operator:

$$\begin{aligned} e^{-\frac{(\varphi+\mathrm{i}\psi)}{h}} h^{2m}\mathcal{L}_\#(x,D)e^{\frac{(\varphi+\mathrm{i}\psi)}{h}} &= (-h^2\Delta - 2hT)^m \\ &+ \sum_{|l|=m-2} e^{-\frac{(\varphi+\mathrm{i}\psi)}{h}} h^{2m-2}\widetilde{c}_l D^l \circ e^{\frac{(\varphi+\mathrm{i}\psi)}{h}}(-h^2\Delta - 2hT) \\ &+ \sum_{|l|\le m-1} e^{-\frac{(\varphi+\mathrm{i}\psi)}{h}} h^{2m} c_l \sum_{k\le l} \frac{l!}{k!(l-k)!} D^k\left(e^{\frac{(\varphi+\mathrm{i}\psi)}{h}}\right) D^{l-k} \end{aligned} \tag{2.12}$$

where

$$T = (\nabla\varphi + \mathrm{i}\nabla\psi)\cdot\nabla + \frac{1}{2}(\Delta\varphi + \mathrm{i}\Delta\psi). \tag{2.13}$$

Now substituting (2.9) in $h^{2m}\mathcal{L}_{\#}(x,D)u=0$, we get the following ransport equations in terms of $a_0, a_1.., a_{m-1}$ by equating the terms involving $h^m, h^{m+1}\cdots h^{2m-1}$ to 0.
Coefficient of h^m :

$$\forall m \geq 2, \quad (-2T)^m a_0 = 0 \quad \text{in } \Omega. \tag{2.14}$$

Notice that since the leading perturbed term is of the form $\sum_{|l|+2=m} \widetilde{c}_l D^l \circ (-\Delta)$, there is no contribution to the coefficient of h^m from this or any other perturbation term.

Coefficient of h^{m+1}:

$$\begin{aligned}(-2T)^m a_1 = &-\sum_{k=0}^{m-1}\Big((-2T)^k \circ (-\Delta)\circ(-2T)^{m-1-k}\Big)a_0\\ &-\sum_{|l|=m-2}\widetilde{c}_l \prod_{\sum_k l_k=m-2}\left(\partial_{x_k}(\varphi+\mathrm{i}\psi)\right)^{l_k}(-2T)a_0\\ &-\sum_{|l|=m-1}c_l\prod_{\sum_k l_k=m-1}\left(\partial_{x_k}(\varphi+\mathrm{i}\psi)\right)^{l_k}a_0.\end{aligned} \tag{2.15}$$

Once a_0 is determined from (2.14), we can determine a_1 by solving (2.15).

In general, the j^{th} transport equation is found by setting the coefficient of h^{m+j} in (2.12) to be 0. It is easy to see that this is an equation of the following form:

$$(-2T)^m a_j = F_j(D)(\widetilde{c}_l, c_l, a_0, \cdots, a_{j-1}) \text{ for } m-j \leq |l| \leq m \tag{2.16}$$

where $F_j(D)$ is a differential operator involving the coefficients $\widetilde{c}_l$ for $|l| = m-2$ and c_l for $m-j \leq |l| \leq m$ together with $(m+j)^{\text{th}}$ derivative of a_0, $(m+j-1)^{\text{th}}$ derivative of $a_1, \cdots$,$(m+1)^{\text{th}}$ derivative of a_{j-1}.

The remainder term $r(x,h)$ satisfies

$$\begin{aligned}&e^{-\frac{(\varphi+i\psi)}{h}}h^{2m}\mathcal{L}_{\#}(x,D)(e^{\frac{(\varphi+i\psi)}{h}}r(x,h))\\ &= -e^{-\frac{(\varphi+i\psi)}{h}}h^{2m}\mathcal{L}_{\#}(x,D)(e^{\frac{(\varphi+i\psi)}{h}}(a_0+ha_1+h^2a_2+..+h^{m-1}a_{m-1}))\\ &= \mathcal{O}(h^{2m}).\end{aligned} \tag{2.17}$$

In order to seek the solutions of these m transport equations (2.16), we follow [9] to transform these equations in cylindrical coordinates. To do so, we apply a translation in $\mathbb{R}^n$ so that $x_0 = 0$ and $\Omega \subset \{x_n > 0\}$, and set $\omega = e_1$. Consider the cylindrical coordinates $(x_1, r\theta)$ on $\mathbb{R}^n$ with $r > 0$ and $\theta \in \mathbb{S}^{n-2}$, and the corresponding change of coordinates is given by the map $x \mapsto (z,\theta)$, where $z = x_1 + ir$ is a complex variable. Then one has

$$\varphi = \mathrm{Re}(\log(z)) \text{ and } \psi = \mathrm{Im}(\log(z)) \text{ or } \varphi + \mathrm{i}\psi = \log(z).$$

Then

$$\begin{aligned}\nabla(\varphi + \mathrm{i}\psi) &= \frac{1}{z}(e_1 + \mathrm{i}e_r), \\ \nabla(\varphi + \mathrm{i}\psi)\cdot\nabla = \frac{2}{z}\partial_{\overline{z}}, \quad \Delta(\varphi+\mathrm{i}\psi) &= -\frac{2(n-2)}{z(z-\overline{z})}\end{aligned} \tag{2.18}$$

where $e_r = (0, \theta), \theta \in \mathbb{S}^{n-2}$. Thus in the cylindrical coordinates the operator T transforms to

$$T = \frac{2}{z}\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right).$$

Let us solve the first transport equation (2.14). We have

$$\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right)^m a_0 = 0 \tag{2.19}$$

There exists $a_0 \in C^\infty(\overline{\Omega})$ satisfying $\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right) a_0 = 0$. Now any solution a_0 to $\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right) a_0 = 0$ is of the form $a_0 = (z-\overline{z})^{(2-n)/2} g_0$ with $g_0 \in C^\infty(\overline{\Omega})$ satisfying $\partial_{\overline{z}} g_0 = 0$.
Similarly the other complex amplitudes can be determined as solutions to the inhomogeneous transport equations:

$$\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right)^m a_j = F_j(D)(\widetilde{c}_l, c_l, a_0, a_1, .., a_{j-1}) := f_j \text{ in } \Omega, 1 \le j \le m-1. \tag{2.20}$$

We have that a_j and f_j have the same regularity. Recall from (2.16), we have that f_j is given in terms of the coefficients $\widetilde{c}_l$ for $|l| = m-2$ and c_l for $m - j \le |l| \le m$ and involves $(m+j)^{\text{th}}$ derivative of $a_0, \cdots, (m+1)^{\text{th}}$ derivative of a_{j-1}. Setting $j = m-1$, we have that $a_{m-1} \in C^{2m}$ since $c_1 \in C^{2m}$. Iteratively proceeding, we have the following:

$$a_{m-1} \in C^{2m}, \cdots, a_{m-k} \in C^{2m+(k-1)(m+1)}, \cdots, a_1 \in C^{2m+(m-2)(m+1)}.$$

Now in order to show the solvability of the inhomogeneous equation (2.20), we break it into a system of m linear equations as follows:

Given $f \in C^{2m}(\overline{\Omega})$, find $v_1 \in C(\overline{\Omega})$ solving $\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right) v_1 = f$ in Ω.

Given v_1, find $v_2 \in C^{2m}(\overline{\Omega})$ solving $\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right) v_2 = v_1$ in Ω.

Proceeding as before, given v_{m-1} find $a_1 \in C^{2m}(\overline{\Omega})$ solving $\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right) a_1 = v_{m-1}$ in Ω.

Now the solvability of the linear inhomogeneous equation

$$\left(\partial_{\overline{z}} - \frac{(n-2)}{2(z-\overline{z})}\right) v = w$$

for $w \in C^r(\overline{\Omega}, \mathbb{C})$ is well known (see [9]), and we get a global solution near $\overline{\Omega}$ with $v \in C^r(\overline{\Omega}, \mathbb{C})$.

Similarly, solvability of the other transport equations can be shown and one obtains the solutions a_j to be at least $C^{2m}(\overline{\Omega})$ for $j = 1, 2.., (m-1)$. In the analysis below, we only require $a_0 \in C^\infty(\overline{\Omega})$. The rest of the amplitudes are chosen so as to have the required decay for the remainder term r. With these choice of amplitudes $a_0, \cdots, a_{m-1}$, we have that

$$e^{-\frac{\varphi}{h}} h^{2m} \mathcal{L}_{\#}(x, D) e^{\frac{\varphi}{h}} (e^{\frac{i\psi}{h}} r(x; h)) = \mathcal{O}(h^{2m}) \text{ in } L^2(\Omega),$$

This equation is solvable by Proposition 2.2 and we have $r \in H^{2m}(\Omega)$ with $||r||_{H^{2m}_{\mathrm{scl}}} = \mathcal{O}(h^m)$. ∎

2.3. ***Boundary Carleman estimate***

Let Ω and φ be as above. We define

$$\partial\Omega_{\pm} = \{x \in \partial\Omega : \pm\partial_\nu \varphi(x) \geq 0\}.$$

Note that due to our choice of φ, $\partial\Omega_-$ is the same as $F(x_0)$ defined in (1.4).

We now derive a Carleman estimate for $\mathcal{L}(x, D)$ involving boundary terms.

Proposition 2.4 *Let for each k, the coefficients $b_k \in C^{2m+(2k-1)(m+1)}(\Omega, \mathbb{C})$, $A_k \in C^{2m+(2k-2)(m+1)}(\Omega, \mathbb{C}^n)$ with $q \in L^\infty(\Omega, \mathbb{C})$ and φ be a limiting Carleman weight for the semiclassical Laplacian on $\widetilde{\Omega}$ with $\Omega \subset\subset \widetilde{\Omega}$. Then for $0 < h \ll 1$, the following boundary Carleman estimate holds for $u \in H^{2m}(\Omega)$, $u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \cdots = (\Delta)^{m-1} u|_{\partial\Omega} = 0$.*

$$\begin{aligned} &\|h^{2m} e^{-\frac{\varphi}{h}} \mathcal{L}(x, D) u\|_{L^2} + \sum_{k=0}^{m-1} h^{\frac{3}{2}+k} \|\sqrt{-\partial_\nu \varphi} e^{-\frac{\varphi}{h}} \partial_\nu \left(-h^2 \Delta\right)^{m-k-1} u\|_{L^2(\partial\Omega_-)} \\ &\geq \sum_{k=0}^{[\![\frac{m}{2}]\!]} h^{m-k} \|e^{-\frac{\varphi}{h}} (-h^2\Delta)^k u\|_{H^1_{\mathrm{scl}}} + \sum_{k=0}^{m-1} h^{\frac{3}{2}+k} \|\sqrt{\partial_\nu \varphi} e^{-\frac{\varphi}{h}} \partial_\nu \left(-h^2 \Delta\right)^{m-k-1} u\|_{L^2(\partial\Omega_+)}. \end{aligned} \tag{2.21}$$

Proof. We start with the boundary Carleman estimates derived in [4, 9] for the semiclassical Laplacian.

$$\begin{aligned} &\|e^{-\frac{\varphi}{h}} (-h^2\Delta) u\|_{L^2} + h^{3/2} \|\sqrt{-\partial_\nu \varphi} e^{-\frac{\varphi}{h}} \partial_\nu u\|_{L^2(\partial\Omega_-)} \geq \frac{1}{C} \Big\{ h \|e^{-\frac{\varphi}{h}} u\|_{H^1_{\mathrm{scl}}} \\ &+ h^{3/2} \|\sqrt{\partial_\nu \varphi} e^{-\frac{\varphi}{h}} \partial_\nu u\|_{L^2(\partial\Omega_+)} \Big\} \end{aligned} \tag{2.22}$$

for $u \in H^2(\Omega), u|_{\partial\Omega} = 0$ and $0 < h \ll 1$. The constant C depends only on Ω and is independent of h. Replacing u with $(-h^2\Delta u)$ for $u \in H^4(\Omega)$ with $u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0$

in the above inequality, we get,

$$
\begin{aligned}
&\|e^{-\frac{\varphi}{h}}(-h^2\Delta)^2 u\|_{L^2} + h^{3/2}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu(-h^2\Delta)u\|_{L^2(\partial\Omega_-)}\\
&\geq \frac{1}{C}\Big\{h\|e^{-\frac{\varphi}{h}}(-h^2\Delta)u\|_{H^1_{\mathrm{scl}}} + h^{3/2}\|\sqrt{\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu(-h^2\Delta)u\|_{L^2(\partial\Omega_+)}\Big\}\\
&= \frac{1}{C}\Big\{\frac{h}{2}\|e^{-\frac{\varphi}{h}}(-h^2\Delta)u\|_{H^1_{\mathrm{scl}}} + \frac{h}{2}\|e^{-\frac{\varphi}{h}}(-h^2\Delta)u\|_{H^1_{\mathrm{scl}}} + h^{3/2}\|\sqrt{\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu(-h^2\Delta)u\|_{L^2(\partial\Omega_+)}\Big\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
&\|e^{-\frac{\varphi}{h}}(-h^2\Delta)^2 u\|_{L^2} + h^{3/2}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu(-h^2\Delta)u\|_{L^2(\partial\Omega_-)} + \frac{h^{5/2}}{2C}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega_-)}\\
&\geq \frac{1}{C}\Bigg\{\frac{h}{2}\|e^{-\frac{\varphi}{h}}(-h^2\Delta)u\|_{H^1_{\mathrm{scl}}} + h^{3/2}\|\sqrt{\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu(-h^2\Delta)u\|_{L^2(\partial\Omega_+)}\\
&+ \frac{h}{2}\Big(\|e^{-\frac{\varphi}{h}}(-h^2\Delta)u\|_{L^2} + h^{3/2}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega_-)}\Big)\Bigg\}.
\end{aligned}
$$

Now using (2.22), we have

$$
\begin{aligned}
&\|e^{-\frac{\varphi}{h}}(-h^2\Delta)^2 u\|_{L^2} + h^{5/2}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega_-)} + h^{3/2}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu(-h^2\Delta)u\|_{L^2(\partial\Omega_-)}\\
&\geq \frac{1}{C}\Big\{h^2\|e^{-\frac{\varphi}{h}}u\|_{H^1_{\mathrm{scl}}} + h\|e^{-\frac{\varphi}{h}}(-h^2\Delta)u\|_{H^1_{\mathrm{scl}}}\\
&+ h^{3/2}\|\sqrt{\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu(-h^2\Delta)u\|_{L^2(\partial\Omega_+)} + h^{5/2}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu u\|_{L^2(\partial\Omega_+)}\Big\}.
\end{aligned}
$$

for $u \in H^4(\Omega), u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0$ and $0 < h \ll 1$. As before, the constant C depends only on Ω and is independent of h.

Iterating this m times, we obtain the following:

$$
\begin{aligned}
&\|e^{-\frac{\varphi}{h}}(-h^2\Delta)^m u\|_{L^2} + \sum_{k=0}^{m-1} h^{3/2+k}\|\sqrt{-\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu\left(-h^2\Delta\right)^{m-k-1}u\|_{L^2(\partial\Omega_-)}\\
&\geq \frac{1}{C}\left(\sum_{k=0}^{[\![\frac{m}{2}]\!]} h^{m-k}\|e^{-\frac{\varphi}{h}}(-h^2\Delta)^k u\|_{H^1_{\mathrm{scl}}} + \sum_{k=0}^{m-1} h^{3/2+k}\|\sqrt{\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu\left(-h^2\Delta\right)^{m-k-1}u\|_{L^2(\partial\Omega_+)}\right).
\end{aligned}
\tag{2.23}
$$

for $u \in H^{2m}(\Omega)$ with $u|_{\partial\Omega} = -\Delta u|_{\partial\Omega} = \cdots = (-\Delta)^{m-1}u = 0$, $0 < h \ll 1$ and C a constant depending on Ω and independent of h.

We now show that we can absorb the lower order terms of (1.1) into this estimate to get the desired estimate (2.21).

Let us consider the following lower order term: $\|e^{-\frac{\varphi}{h}}h^{2m-2k}b_k(-h^2\Delta)^k u\|_{L^2}$. This can be estimated as

$$
\|e^{-\frac{\varphi}{h}}h^{2m-2k}b_k(-h^2\Delta)^k u\|_{L^2} \leq \mathcal{O}(h^{m-k})h^{m-k}\|e^{-\frac{\varphi}{h}}(-h^2\Delta)^k u\|_{H^1_{\mathrm{scl}}}.
$$

Since k is at most $[\![m/2]\!]$, all such terms can be absorbed to the right hand side of (2.23). Next we consider the term $\|e^{-\frac{\varphi}{h}}h^{2m}\left(A_k\cdot D\right)(-\Delta)^{k-1}u\|_{L^2}$. Letting $v =$

$(-\Delta)^k u$, from [10], we have the following:

$$\|e^{-\frac{\varphi}{h}} h^{2m} A\cdot Dv\|_{L^2} = \mathcal{O}(h^{2m-1})\|e^{-\frac{\varphi}{h}} v\|_{H^1_{\mathrm{scl}}}.$$

Using this, we get

$$\|e^{-\frac{\varphi}{h}} h^{2m} \left(A_k\cdot D\right)\left(-\Delta\right)^{k-1} u\|_{L^2} \leq \mathcal{O}(h^{m-k}) h^{m-k+1}\|e^{-\frac{\varphi}{h}}(-h^2\Delta)^{k-1}u\|_{H^1_{\mathrm{scl}}}.$$

As before, since k is at most $[\![m/2]\!]$, all such terms can be absorbed to the right hand side of (2.23).

This completes the proof of the boundary Carleman estimate (2.21). ■

Remark 1 For the interior and boundary Carleman estimates, it is enough to work with $A_k \in W^{2k-1,\infty}(\Omega,\mathbb{C}^n)$, $b_k \in W^{2k,\infty}(\Omega,\mathbb{C})$ and $q\in L^\infty(\Omega)$.

3. Determination of the coefficients

3.1. *Integral identity involving the coefficients* $\boldsymbol{b_k, A_k, q}$

We recall that

$$\mathcal{L}(x,D) = (-\Delta)^m + \sum_{k=1}^{[\![\frac{m}{2}]\!]} b_k(x)(-\Delta)^k + \sum_{k=1}^{[\![\frac{m+1}{2}]\!]} (A_k(x)\cdot D)(-\Delta)^{k-1} + q(x).$$

where, for each k, $b_k \in C^{2m+(2k-1)(m+1)}(\overline{\Omega},\mathbb{C})$, $A_k \in C^{2m+(2k-2)(m+1)}(\overline{\Omega},\mathbb{C}^n)$ and $q\in L^\infty(\overline{\Omega},\mathbb{C})$.

We also recall that the formal L^2 adjoint of this operator, $\mathcal{L}^*(x,D)$, is of the form

$$\mathcal{L}^*(x,D) \equiv \begin{cases} (-\Delta)^m + \overline{b_{\frac{m}{2}}}(x)(-\Delta)^{\frac{m}{2}} + \sum\limits_{|l|\leq m-1} c_l D^l, \text{ for } m \text{ even;}\\ (-\Delta)^m + (\overline{A_{\frac{m+1}{2}}(x)}\cdot D)(-\Delta)^{\frac{m-1}{2}} + \sum\limits_{|l|\leq m-1} c_l D^l, \text{ for } m \text{ odd.}\end{cases} \tag{3.1}$$

where the coefficients $c_l \in C^{2m+(|l|-1)(m+1)}(\overline{\Omega};\mathbb{C})$.

We now derive an integral identity involving the coefficients b_k, A_k and q.

Proposition 3.1

$$\int_\Omega (\mathcal{L}(x,D)u)\overline{v}\mathrm{d}x - \int_\Omega u\overline{\mathcal{L}^*(x,D)v}\mathrm{d}x = I + II + III \tag{3.2}$$

where I, II *and* III *are as follows:*

$$I = \sum_{l=1}^m \int_{\partial\Omega} \Big((-\Delta)^{m-l}u(\overline{\partial_\nu(-\Delta)^{l-1}v}) - (\partial_\nu((-\Delta)^{m-l}u)(\overline{(-\Delta)^{l-1}v})\Big)\,\mathrm{d}\sigma.$$

$$II = \sum_{k=1}^{[\![\frac{m}{2}]\!]}\sum_{l=1}^k \int_{\partial\Omega}\Big(((-\Delta)^{k-l}u)\overline{\partial_\nu\left((-\Delta)^{l-1}\left(\overline{b_k}v\right)\right)} - (\partial_\nu((-\Delta)^{k-l}u)(\overline{(-\Delta)^{l-1}(\overline{b_k}v)})\Big)\mathrm{d}\sigma.$$

$$III=\sum_{k=2}^{[\![\frac{m+1}{2}]\!]}\sum_{l=1}^{k-1}\int_{\partial\Omega}\Big(((-\Delta)^{k-1-l}u)(\overline{\partial_\nu\left((-\Delta)^{l-1}(D\cdot(\overline{A_k}v))\right)})$$

$$-\left(\partial_\nu((-\Delta)^{k-1-l}u\right)\overline{((-\Delta)^{l-1}(D\cdot(\overline{A_k}v))}\Big)\mathrm{d}\sigma+\frac{1}{\mathrm{i}}\sum_{k=1}^{[\![\frac{m+1}{2}]\!]}\int\limits_{\partial\Omega}(-\Delta)^{k-1}u\overline{v}A_k\cdot\nu\mathrm{d}\sigma.$$

Here as usual $\nu(x)$ is the unit outer normal vector on the boundary and $d\sigma$ is the boundary surface measure.

Proof. Let us consider the case of m even. The proof for odd m is similar. We have

$$\int_\Omega(\mathcal{L}(x,D)u)\overline{v}\mathrm{d}x-\int_\Omega u\overline{\mathcal{L}^*(x,D)v}\mathrm{d}x=\int\limits_\Omega(-\Delta)^m u\overline{v}\,\mathrm{d}x-\int\limits_\Omega u\overline{(-\Delta)^m v}\,\mathrm{d}x$$

$$+\sum_{k=1}^{m/2}\int\limits_\Omega b_k(-\Delta)^k u\overline{v}\,\mathrm{d}x+\sum_{k=1}^{m/2}\int\limits_\Omega(A_k\cdot D)(-\Delta)^{k-1}u\overline{v}\,\mathrm{d}x$$

$$-\int\limits_\Omega b_{\frac{m}{2}}u\overline{(-\Delta)^{m/2}v}\,\mathrm{d}x-\int\limits_\Omega u\overline{\sum_{|l|\le m-1}c_lD^lv}\,\mathrm{d}x.$$

By Green formula, we have

$$\int\limits_\Omega(-\Delta)^m u\overline{v}\,\mathrm{d}x-\int\limits_\Omega(-\Delta)^{m-1}u\overline{(-\Delta v)}\mathrm{d}x=\int\limits_{\partial\Omega}\left((-\Delta)^{m-1}u\frac{\partial v}{\partial\nu}-\frac{\partial}{\partial\nu}\left((-\Delta)^{m-1}u\right)\overline{v}\right)\mathrm{d}\sigma.$$

Now repeated use of this formula gives

$$\int\limits_\Omega(-\Delta)^m u\overline{v}\,\mathrm{d}x-\int\limits_\Omega u\overline{(-\Delta)^m v}\,\mathrm{d}x$$

$$=\sum_{l=1}^{m}\int_{\partial\Omega}\left(((-\Delta)^{m-l}u)(\overline{\partial_\nu(-\Delta)^{l-1}v})-(\partial_\nu((-\Delta)^{m-l}u)(\overline{(-\Delta)^{l-1}v})\right)\mathrm{d}\sigma.$$

Similarly, repeated use of Green formula gives

$$
\begin{aligned}
&\sum_{k=1}^{\frac{m}{2}} \int b_k \left(-\Delta\right)^k u\overline{v}\, \mathrm{d}x + \sum_{k=1}^{\frac{m}{2}} \int \left(A_k \cdot D\right)\left(-\Delta\right)^{k-1} u\overline{v}\, \mathrm{d}x \\
&- \int b_{\frac{m}{2}} u\overline{(-\Delta)^{m/2}\, v}\, \mathrm{d}x - \int u \overline{\sum_{|l|\le m-1} c_l D^l v}\, \mathrm{d}x \\
&= \sum_{k=1}^{\frac{m}{2}} \sum_{l=1}^{k} \int_{\partial\Omega} \Big(((-\Delta)^{k-l} u)\overline{\partial_\nu \left((-\Delta)^{l-1}\left(\overline{b_k} v\right)\right)} \\
&- (\partial_\nu((-\Delta)^{k-l} u)(\overline{(-\Delta)^{l-1}(\overline{b_k} v)})\Big)\mathrm{d}\sigma \\
&+ \sum_{k=2}^{\frac{m}{2}} \sum_{l=1}^{k-1} \int_{\partial\Omega} \Big(((-\Delta)^{k-1-l} u)(\overline{\partial_\nu \left((-\Delta)^{l-1}(D \cdot (\overline{A_k} v)\right)}) \\
&- \left(\partial_\nu((-\Delta)^{k-1-l} u\right)(\overline{(-\Delta)^{l-1}(D \cdot (\overline{A_k} v))})\Big)\mathrm{d}\sigma \\
&+ \frac{1}{\mathrm{i}} \sum_{k=1}^{m/2} \int_{\partial\Omega} (-\Delta)^{k-1} u\overline{v} A_k \cdot \nu\, \mathrm{d}\sigma.
\end{aligned}
$$

■

Lemma 3.2 *Consider two operators* $\mathcal{L}(x, D)$ *and* $\widetilde{\mathcal{L}}(x, D)$ *as in Theorem 1.1 and assume that* u *and* $\widetilde{u}$ *are the solutions respectively of the boundary value problem (1.2) corresponding to* $\mathcal{L}$ *and* $\widetilde{\mathcal{L}}$ *respectively. Suppose in the open neighborhood* $F^\#$ *of* $\partial\Omega_- \subset \partial\Omega$*, we have*

$$
\partial_\nu(-\Delta)^{l-1} u|_{F^\#} = \partial_\nu(-\Delta)^{l-1}\widetilde{u}|_{F^\#} \text{ for } l = 1, 2, \cdots, m-1. \tag{3.3}
$$

If $v \in H^{2m}(\Omega)$ *satisfies* $\mathcal{L}^*(x, D)v = 0$ *where* $\mathcal{L}^*(x, D)$ *is the formal* L^2 *adjoint of* $\mathcal{L}(x, D)$*, then we have the following integral identity:*

$$
\begin{aligned}
&\int_\Omega \left(\sum_{k=1}^{[\![\frac{m}{2}]\!]} (b_k - \widetilde{b_k})(-\Delta)^k \widetilde{u}\overline{v} + \sum_{k=1}^{[\![\frac{m+1}{2}]\!]} ((A_k - \widetilde{A}_k) \cdot D)(-\Delta)^{k-1} \widetilde{u}\overline{v} + (q - \widetilde{q})\widetilde{u}\overline{v} \right) \mathrm{d}x \\
&= \sum_{l=1}^{m} \left(\int_{\partial\Omega\setminus F^\#} (\partial_\nu((-\Delta)^{m-l}(u - \widetilde{u})(\overline{(-\Delta)^{l-1} v}) \mathrm{d}\sigma \right) \\
&+ \sum_{k=1}^{[\![\frac{m}{2}]\!]} \sum_{l=1}^{k} \int_{\partial\Omega\setminus F^\#} [(\partial_\nu((-\Delta)^{k-l}(u - \widetilde{u})(\overline{(-\Delta)^{l-1}(\overline{b_k} v)}) \mathrm{d}\sigma \\
&+ \sum_{k=2}^{[\![\frac{m+1}{2}]\!]} \sum_{l=1}^{k-1} \int_{\partial\Omega\setminus F^\#} [(\partial_\nu((-\Delta)^{k-1-l}(u - \widetilde{u})(\overline{(-\Delta)^{l-1}(D \cdot (\overline{A_k} v))}) \mathrm{d}\sigma \\
&+ \frac{1}{\mathrm{i}} \sum_{k=1}^{[\![\frac{m+1}{2}]\!]} \int_{\partial\Omega} (-\Delta)^{k-1}(u - \widetilde{u})\overline{v} A_k \cdot \nu\, \mathrm{d}\sigma.
\end{aligned}
$$

Proof. We have

$$\mathcal{L}(x,D)(u-\widetilde{u}) = \sum_{k=1}^{[\![\frac{m}{2}]\!]}(b_k-\widetilde{b}_k)(-\Delta)^k\widetilde{u}+\sum_{k=1}^{[\![\frac{m+1}{2}]\!]}((A_k-\widetilde{A}_k)\cdot D)(-\Delta)^{k-1}\widetilde{u}+(q-\widetilde{q})\widetilde{u}. \quad (3.4)$$

Now using Proposition 3.1, this lemma follows. ■

Now let v and $\widetilde{u}$ be CGO solutions of $\mathcal{L}^*(x,D)v = 0$ and $\widetilde{\mathcal{L}}(x,D)\widetilde{u} = 0$, respectively, of the form:

$$v(x;h) = e^{\frac{(\varphi_1+\mathrm{i}\psi_1)}{h}}(a_0^{(1)}+ha_1^{(1)}+h^2a_2^{(1)}+\cdots+h^{m-1}a_{m-1}^{(1)}+r^{(1)}(x;h)) \in H^{2m}(\Omega) \quad (3.5)$$

$$\widetilde{u}(x,h) = e^{\frac{(\varphi_2+\mathrm{i}\psi_2)}{h}}(a_0^{(2)}+ha_1^{(2)}+h^2a_2^{(2)}+\cdots+h^{m-1}a_{m-1}^{(2)}+r^{(2)}(x;h)) \in H^{2m}(\Omega) \quad (3.6)$$

where

$$-\varphi_1(x) = \varphi_2(x) = \varphi(x) = \log|x-x_0|, \;\; \psi_1(x) = \psi_2(x) = d_{\mathbb{S}^{n-1}}\left(\frac{x-x_0}{|x-x_0|},\omega\right).$$

LEMMA 3.3 *Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be as above with $u,\widetilde{u} \in H^{2m}(\Omega)$ being their solutions respectively with $\widetilde{u}$ given by* (3.6) *and v given by* (3.5) *being a solution of $\mathcal{L}^*v = 0$. Additionally suppose $b_l = \widetilde{b}_l$ for $k+1 \le l \le [\![\frac{m}{2}]\!]$ and $A_l = \widetilde{A}_l$ for $k+1 \le l \le [\![\frac{m+1}{2}]\!]$. Then*

$$\begin{aligned} h^k\Bigg\{ &\sum_{l=1}^{m}\left(\int_{\partial\Omega\setminus F^\#}(\partial_\nu((-\Delta)^{m-l}(u-\widetilde{u})(\overline{(-\Delta)^{l-1}v})\right)\mathrm{d}\sigma \\ &+\sum_{k=1}^{[\![\frac{m}{2}]\!]}\sum_{l=1}^{k}\int_{\partial\Omega\setminus F^\#}\Big((\partial_\nu((-\Delta)^{k-l}(u-\widetilde{u}))(\overline{(-\Delta)^{l-1}(\overline{b_k}v)})\Big)\mathrm{d}\sigma \\ &+\sum_{k=2}^{[\![\frac{m+1}{2}]\!]}\sum_{l=1}^{k-1}\int_{\partial\Omega\setminus F^\#}\Big((\partial_\nu((-\Delta)^{k-l-1}(u-\widetilde{u})))(\overline{(-\Delta)^{l-1}(D\cdot\overline{A_k}v)})\Big)\mathrm{d}\sigma \\ &+\frac{1}{\mathrm{i}}\sum_{k=1}^{[\![\frac{m+1}{2}]\!]}\int_{\partial\Omega}(-\Delta)^{k-1}(u-\widetilde{u})\overline{v}A_k\cdot\nu\,\mathrm{d}\sigma\Bigg\} \to 0 \end{aligned} \quad (3.7)$$

as $h \to 0$.

Proof. Choose $\varepsilon > 0$ small enough such that

$$\partial\Omega_- \subset F_\varepsilon = \{x \in \partial\Omega : \partial_\nu\varphi \le \varepsilon\} \subset F^\#.$$

Let us consider

$$\left|\int_{\partial\Omega\setminus F^\#}\partial_\nu((-\Delta)^{m-l}(u-\widetilde{u})(\overline{(-\Delta)^{l-1}v}))\mathrm{d}\sigma\right|$$

We have

$$\begin{aligned}
&\left|\int_{\partial\Omega\setminus F^{\#}} \partial_\nu((-\Delta)^{m-l}(u-\widetilde{u})(\overline{(-\Delta)^{l-1}v}))\mathrm{d}\sigma\right| \\
&\leq \int_{\partial\Omega\setminus F_\varepsilon} \left|\partial_\nu((-\Delta)^{m-l}(u-\widetilde{u}))e^{-\frac{\varphi}{h}}\right| \left|(-\Delta+\frac{2}{h}\overline{T})^{l-1}(a_0^{(1)}(x)+\cdots+r^{(1)}(x;h))\right|\mathrm{d}\sigma \\
&\leq \frac{1}{\sqrt{\varepsilon}}\left(\int_{\partial\Omega\setminus F_\varepsilon} \epsilon|\partial_\nu((-\Delta)^{m-l}(u-\widetilde{u}))|^2 e^{-\frac{2\varphi}{h}}\mathrm{d}\sigma\right)^{1/2} \\
&\times \|(-\Delta+\frac{2}{h}\overline{T})^{l-1}(a_0^{(1)}(x)+\cdots+r^{(1)}(x;h))\|_{L^2(\partial\Omega)} \\
&\leq \|\sqrt{\partial_\nu\phi}e^{-\frac{\phi}{h}}\partial_\nu((-\Delta)^{m-l}(u-\widetilde{u}))\|_{L^2(\partial\Omega_+)}\|(-\Delta+\frac{2}{h}\overline{T})^{l-1}(a_0^{(1)}(x)+\cdots+r^{(1)}(x;h))\|_{L^2(\partial\Omega)}.
\end{aligned}$$

Since $\|r^{(1)}\|_{H^{2m}_{\mathrm{scl}}} = \mathcal{O}(h^m)$, we have

$$\|r^{(1)}\|_{H^m(\Omega)} = \mathcal{O}(1) \text{ and } \|r^{(1)}\|_{H^{m+j}(\Omega)} = \mathcal{O}(\frac{1}{h^j}) \text{ for } 1\leq j\leq m.$$

Then

$$\begin{aligned}
&\partial^\alpha r^{(1)}|_{\partial\Omega} = \mathcal{O}(1) \text{ in } L^2(\partial\Omega) \text{ for } |\alpha|\leq m-1 \text{ and} \\
&\partial^\alpha r^{(1)}|_{\partial\Omega} = \mathcal{O}(\frac{1}{h^j}) \text{ in } L^2(\partial\Omega) \text{ for } |\alpha| = m-1+j, 1\leq j\leq m.
\end{aligned}$$

We have

$$(-\Delta)^{l-1}r^{(1)}|_{\partial\Omega} \text{ in } L^2(\partial\Omega) = \begin{cases}\mathcal{O}(1) \text{ if } 2l\leq m+1 \\ \mathcal{O}(\frac{1}{h^{2l-m-1}}) \text{ if } 2l > m+1.\end{cases}$$

Now

$$(-\Delta)^{l-1}v = e^{\frac{-\varphi+i\psi}{h}}(-\Delta+\frac{2}{h}\overline{T})^{l-1}(a_0^{(1)}(x)+ha_1^{(1)}(x)+\cdots+r^{(1)}(x;h))$$

Then for $0<h\ll 1$ and $m\geq 2$,

$$\begin{aligned}
(-\Delta)^{l-1}v|_{\partial\Omega} \text{ in } L^2(\partial\Omega) &= \mathcal{O}(\frac{1}{h^{l-1}}) + \text{Order of } (-\Delta)^{l-1}r^{(1)}|_{\partial\Omega} \\
&= \mathcal{O}(\frac{1}{h^{l-1}}) \quad \text{as } l-1\geq 2l-m-1 \text{ for } \quad 1\leq l\leq m.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\left|\int_{\partial\Omega\setminus F^{\#}} \partial_\nu((-\Delta)^{m-l}(u-\widetilde{u})(\overline{(-\Delta)^{l-1}v}))\mathrm{d}\sigma\right| \\
&= \mathcal{O}(h^{1-l})\|\sqrt{\partial_\nu\varphi}e^{-\frac{\varphi}{h}}\partial_\nu((-\Delta)^{m-l}(u-\widetilde{u}))\|_{L^2(\partial\Omega_+)}.
\end{aligned}$$

Now applying the boundary Carleman estimate, we have

$$\|\sqrt{\partial_\nu \varphi} e^{-\frac{\varphi}{h}} \partial_\nu ((-\Delta)^{m-l}(u-\widetilde{u}))\|_{L^2(\partial\Omega_+)} \leq \frac{C}{h^{\frac{1}{2}-l}} \|e^{-\frac{\varphi}{h}} \mathcal{L}(x,D)(u-\widetilde{u})\|_{L^2}$$

Hence

$$\left| \int_{\partial\Omega\setminus F^\#} \partial_\nu ((-\Delta)^{m-l}(u-\widetilde{u})(\overline{(-\Delta)^{l-1} v})) \mathrm{d}\sigma \right| \leq \mathcal{O}(h^{\frac{1}{2}}) \|e^{-\frac{\varphi}{h}} \mathcal{L}(x,D)(u-\widetilde{u})\|_{L^2}.$$

We have

$$e^{-\frac{\varphi}{h}} \mathcal{L}(x,D)(u-\widetilde{u}) = e^{-\frac{\varphi}{h}} \sum_{l=1}^{k} (b_l - \widetilde{b}_l)(-\Delta)^l \widetilde{u} + e^{-\frac{\varphi}{h}} \sum_{l=1}^{k} \left((A_l - \widetilde{A}_l) \cdot D \right) (-\Delta)^{l-1} \widetilde{u} + e^{-\frac{\varphi}{h}} (q - \widetilde{q}) \widetilde{u},$$

since $b_l = \widetilde{b}_l$ for $k+1 \leq l \leq [\![\frac{m}{2}]\!]$ and $A_l = \widetilde{A}_l$ for $k+1 \leq l \leq [\![\frac{m+1}{2}]\!]$.
Using the solution $\widetilde{u}$ given in (3.6) in the above equation, we have,

$$\begin{aligned} e^{-\frac{\varphi}{h}} \mathcal{L}(x,D)(u-\widetilde{u}) &= e^{\frac{\mathrm{i}\psi_2}{h}} \left(\sum_{l=1}^{k} (b_l - \widetilde{b}_l) \left(-\Delta - \frac{2T}{h} \right)^l \right. \\ &+ \sum_{l=1}^{k} \left((A_l - \widetilde{A}_l) \cdot \frac{1}{h} D(\varphi_2 + \mathrm{i}\psi_2) \right) \left(-\Delta - \frac{2T}{h} \right)^{l-1} \\ &+ \left. \sum_{l=1}^{k} ((A_l - \widetilde{A}_l) \cdot D \left(-\Delta - \frac{2T}{h} \right)^{l-1} + (q - \widetilde{q}) \right) \left(a_0^{(2)} + h a_1^{(2)} + \cdots + r^{(2)} \right). \end{aligned}$$

Now from the above expression, we have that

$$\|e^{-\frac{\varphi}{h}} \mathcal{L}(x,D)(u-\widetilde{u})\|_{L^2(\Omega)} \leq \mathcal{O}\left(\frac{1}{h^k} \right).$$

Therefore we have

$$\left| \int_{\partial\Omega\setminus F^\#} \partial_\nu ((-\Delta)^{m-l}(u-\widetilde{u})(\overline{(-\Delta)^{l-1} v})) \mathrm{d}\sigma \right| \leq \mathcal{O}(h^{\frac{1}{2}}) \|e^{-\frac{\varphi}{h}} \mathcal{L}(x,D)(u-\widetilde{u})\|_{L^2} \quad \leq \mathcal{O}\left(\frac{h^{1/2}}{h^k} \right).$$

Applying the same argument, we can show that the remaining terms in (3.7) are terms of order $\mathcal{O}\left(\frac{h^{1/2}}{h^k}\right)$.

Hence

$$h^k\Bigg\{\sum_{l=1}^{m}\left(\int_{\partial\Omega\setminus F^{\#}}(\partial_\nu((-\Delta)^{m-l}(u-\widetilde{u})(\overline{(-\Delta)^{l-1}v})\right)\mathrm{d}\sigma$$

$$+\sum_{k=1}^{[\![\frac{m}{2}]\!]}\sum_{l=1}^{k}\int_{\partial\Omega\setminus F^{\#}}\Big((\partial_\nu((-\Delta)^{k-l}(u-\widetilde{u}))(\overline{(-\Delta)^{l-1}(\overline{b_k}v)})\Big)\,\mathrm{d}\sigma$$

$$+\sum_{k=2}^{[\![\frac{m+1}{2}]\!]}\sum_{l=1}^{k-1}\int_{\partial\Omega\setminus F^{\#}}\Big((\partial_\nu((-\Delta)^{k-l-1}(u-\widetilde{u})))(\overline{(-\Delta)^{l-1}(D\cdot\overline{A_k}v)})\Big)\,\mathrm{d}\sigma$$

$$+\frac{1}{\mathrm{i}}\sum_{k=1}^{[\![\frac{m+1}{2}]\!]}\int_{\partial\Omega}(-\Delta)^{k-1}(u-\widetilde{u})\overline{v}A_k\cdot\nu\,\mathrm{d}\sigma\Bigg\}\le\mathcal{O}\left(h^{1/2}\right)\to 0\text{ as }h\to 0.$$

■

Now under the assumptions of Lemma 3.3, we have

$$h^k\int_\Omega\left(\sum_{l=1}^{k}(b_l-\widetilde{b}_l)(-\Delta)^l\widetilde{u}\overline{v}+\sum_{l=1}^{k}((A_l-\widetilde{A}_l)\cdot D)(-\Delta)^{l-1}\widetilde{u}\overline{v}+(q-\widetilde{q})\widetilde{u}\overline{v}\right)\mathrm{d}x\to 0\text{ as }h\to 0.$$

Substituting the solutions (3.6) and (3.5) for $\widetilde{u}$ and $\widetilde{v}$, respectively, in the above integral, we get

$$h^k\int_\Omega\sum_{l=1}^{k}(b_l-\widetilde{b}_l)\left(-\Delta-\frac{2}{h}T\right)^l\left(a_0^{(2)}+\cdots+r^{(2)}\right)\overline{\left(a_0^{(1)}+\cdots+r^{(1)}\right)}\,\mathrm{d}x$$

$$+\int_\Omega\sum_{l=1}^{k}((A_l-\widetilde{A}_l)\cdot\left(D\left(-\Delta-\frac{2}{h}T\right)^{l-1}\left(a_0^{(2)}+\cdots+r^{(2)}\right)\right)\overline{\left(a_0^{(1)}+\cdots+r^{(1)}\right)}\mathrm{d}x$$

$$+\int_\Omega\sum_{l=1}^{k}\left((A_l-\widetilde{A}_l)\cdot\frac{D(\varphi+\mathrm{i}\psi)}{h}\right)\left(-\Delta-\frac{2}{h}T\right)^{l-1}\left(a_0^{(2)}+\cdots+r^{(2)}\right)\overline{\left(a_0^{(1)}+\cdots+r^{(1)}\right)}\mathrm{d}x$$

$$+\int_\Omega(q-\widetilde{q})\left(a_0^{(2)}+\cdots+r^{(2)}\right)\overline{\left(a_0^{(1)}+\cdots+r^{(1)}\right)}\mathrm{d}x\to 0\text{ as }h\to 0.$$

Taking limits as $h\to 0$, we arrive at the following integral identity.

Lemma 3.4 *Under the assumptions of Lemma 3.3, we have the following integral identity:*

$$\int_\Omega\Big\{\left(b_k-\widetilde{b}_k\right)\left((-2T)^k(a_0^{(2)})\right)\overline{a_0^{(1)}}+((A_k-\widetilde{A}_k)\cdot D(\varphi+\mathrm{i}\psi))\left((-2T)^{k-1}(a_0^{(2)})\right)\overline{a_0^{(1)}}\Big\}\mathrm{d}x=0.$$

Remark 2 If we do not assume the equality of the coefficients $b_l=\widetilde{b}_l$ for $k+1\le l\le[\![\frac{m}{2}]\!]$ and $A_l=\widetilde{A}_l$ for $k+1\le l\le[\![\frac{m+1}{2}]\!]$, then from the same arguments as above we will arrive at the following integral identities:

(1) For m even:

$$\int_\Omega \left\{ \left(b_{\frac{m}{2}} - \widetilde{b}_{\frac{m}{2}}\right) \left((-2T)^{\frac{m}{2}}(a_0^{(2))}\right) \overline{a_0^{(1)}} \right. \\ \left. + ((A_{\frac{m}{2}} - \widetilde{A}_{\frac{m}{2}}) \cdot D(\varphi + \mathrm{i}\psi)) \left((-2T)^{\frac{m}{2}-1}(a_0^{(2)})\right) \overline{a_0^{(1)}} \right\} \mathrm{d}x = 0. \tag{3.8}$$

(2) For m odd:

$$\int_\Omega \left(\left(A_{\frac{m+1}{2}} - \widetilde{A}_{\frac{m+1}{2}}\right) \cdot D(\phi + \mathrm{i}\psi)\right) \left((-2T)^{\frac{m-1}{2}}(a_0^{(2)})\right) \overline{a_0^{(1)}} \mathrm{d}x = 0. \tag{3.9}$$

3.2. *Uniqueness of the coefficients*

In this section, we will show, under the assumption of equality of the coefficients $b_l = \widetilde{b}_l$ for $k+1 \leq l \leq [\![\frac{m}{2}]\!]$ and $A_l = \widetilde{A}_l$ for $k+1 \leq l \leq [\![\frac{m+1}{2}]\!]$, that $b_k = \widetilde{b}_k$ using the integral identity of Lemma 3.4.

The important idea is to use the degree of freedom one has in choosing amplitudes $a_0^{(2)}$ satisfying (2.19) and $a_0^{(1)}$ satisfying $\overline{T}^m(a_0^{(1)}) = 0$ in order to achieve the goal. This will complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We recall the following integral identity:

$$\int_\Omega \left\{ (b_k - \widetilde{b}_k)(-2T)^k(a_0^{(2)})\overline{a_0^{(1)}} + \left((A_k - \widetilde{A}_k) \cdot D(\varphi + \mathrm{i}\psi)\right)(-2T)^{k-1}(a_0^{(2)})\overline{a_0^{(1)}} \right\} \mathrm{d}x = 0 \tag{3.10}$$

for all $a_0^{(2)}, a_0^{(1)} \in C^\infty(\overline{\Omega})$ satisfying the transport equations:

$$T^m(a_0^{(2)}) = 0 \text{ and } \overline{T}^m(a_0^{(1)}) = 0,$$

where

$$T = (\nabla\varphi + \mathrm{i}\nabla\psi) \cdot \nabla + \frac{(\Delta\varphi + \mathrm{i}\Delta\psi)}{2}.$$

We seek $a_0^{(1)} = e^{\Phi_1}$ with $\Phi_1 \in C^\infty(\overline{\Omega})$, and $a_0^{(2)} \in C^\infty(\overline{\Omega})$ solving

$$T^{k-1}(a_0^{(2)}) = ge^{\Phi_2}$$

with $g, \Phi_2 \in C^\infty(\overline{\Omega})$ such that $T^k(a_0^{(2)}) = 0$. Subsequently we have $T^m(a_0^{(2)}) = 0$.

Also we seek $a_0^{(1)}$ such that $\overline{T}(a_0^{(1)}) = 0$. This then implies that $\overline{T}^m(a_0^{(1)}) = 0$.

In other words, we seek g, Φ_1 and Φ_2 such that

$$(\nabla\varphi + \mathrm{i}\psi) \cdot \nabla g = 0, \quad (\nabla\varphi + \mathrm{i}\nabla\psi) \cdot \nabla\overline{\Phi_1} + \frac{(\Delta\varphi + \mathrm{i}\Delta\psi)}{2} = 0,$$

$$(\nabla\varphi + \mathrm{i}\nabla\psi) \cdot \nabla\Phi_2 + \frac{(\Delta\varphi + \mathrm{i}\Delta\psi)}{2} = 0.$$

Such choices of $a_0^{(1)}$ and $a_0^{(2)}$ are possible as follows from [10, Equations 5.2 – 5.4]. Now by putting these particular choices of $a_0^{(1)}, a_0^{(2)}$ in (3.10), we get the following integral identity involving only $(A_k - \widetilde{A}_k)$:

$$\int_\Omega (A_k - \widetilde{A}_k) \cdot \nabla(\varphi + \mathrm{i}\psi) g e^{\Phi_2 + \overline{\Phi_1}} \mathrm{d}x = 0.$$

Now proceeding exactly as in [10, Section 5], we conclude that

$$A_k = \widetilde{A}_k \text{ in } \Omega. \tag{3.11}$$

Our next step is to show that $b_k = \widetilde{b}_k$ in Ω. Now we have

$$\int_\Omega (b_k - \widetilde{b}_k) T^k(a_0^{(2)}) \overline{a_0^{(1)}} = 0 \tag{3.12}$$

Now we simply seek $a_0^{(1)} = e^{\Phi_1}$ with $\Phi_1 \in C^\infty(\overline{\Omega})$ such that $\overline{T}(a_0^{(1)}) = 0$ (then $\overline{T}^m(a_0^{(1)}) = 0$) and $a_0^{(2)} \in C^\infty(\overline{\Omega})$ solving $T^k(a_0^{(2)}) = ge^{\Phi_2}$ with $g, \Phi_2 \in C^\infty(\overline{\Omega})$ such that

$$T^{k+1}(a_0^{(2)}) = 0.$$

Subsequently we have $T^m(a_0^{(2)}) = 0$ for $m \geq 2$. Thus we obtain,

$$\int_\Omega (b_k - \widetilde{b}_k) g e^{\Phi_2 + \overline{\Phi_1}} = 0. \tag{3.13}$$

Again proceeding by the same analysis from [4], we obtain

$$b_k = \widetilde{b}_k \text{ in } \Omega. \tag{3.14}$$

Proceeding similarly, we can show that $b_k = \widetilde{b}_k$ for all $1 \leq k \leq [\![\frac{m}{2}]\!]$ and $A_k = \widetilde{A}_k$ for $1 \leq k \leq [\![\frac{m+1}{2}]\!]$.

Finally we will show the unique recovery of the zeroth order perturbed coefficient. Now that we have established that $b_k = \widetilde{b}_k$ for $1 \leq k \leq [\![\frac{m}{2}]\!]$ and $A_k = \widetilde{A}_k$ for all $1 \leq k \leq [\![\frac{m+1}{2}]\!]$, taking limit as $h \to 0$, we have the following integral identity:

$$\int_\Omega (q - \widetilde{q}) a_0^{(2)} \overline{a_0^{(1)}} \mathrm{d}x = 0. \tag{3.15}$$

Now to show $q = \widetilde{q}$ from (3.15) we choose $a_0^{(2)} = ge^{\Phi_2}$ where $\Phi_2 \in C^\infty(\overline{\Omega})$ is such that $T(e^{\Phi_2}) = 0$ and $g \in C^\infty(\overline{\Omega})$ is such that $(\nabla\phi + \mathrm{i}\nabla\psi) \cdot \nabla g = 0$, so that it satisfies $T^m(a_0^{(2)}) = 0$. Also we seek $a_0^{(1)} = e^{\Phi_1}$, with $\Phi_1 \in C^\infty(\overline{\Omega})$ which satisfies $\overline{T} e^{\Phi_1} = 0$, so that $\overline{T}^m a_0^{(1)} = 0$. Then (3.15) gives,

$$\int_\Omega (q - \widetilde{q}) g e^{\overline{\Phi_1} + \Phi_2} \mathrm{d}x = 0. \tag{3.16}$$

Again following the argument from [4, Section 6], from the above identity we conclude that

$$q = \widetilde{q} \text{ in } \Omega.$$

This completes the proof of Theorem 1.1. ■

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