An inverse problem for the relativistic Schrödinger equation with partial boundary data

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ABSTRACT
We study the inverse problem of determining the vector and scalar potentials $\mathcal{A} = (A_0(t,x), A_1(t,x), \cdots, A_n(t,x))$ and $q(t,x)$, respectively, in the relativistic Schrödinger equation

$$\Big((\partial_t + A_0(t,x))^2 - \sum_{j=1}^{n} (\partial_j + A_j(t,x))^2 + q(t,x) \Big) u(t,x) = 0$$

in the region $Q = (0,T) \times \Omega$, where Ω is a C^2 bounded domain in $\mathbb{R}^n$ for $n \geq 3$ and $T > \operatorname{diam}(\Omega)$ from partial data on the boundary ∂Q. We prove the unique determination of these potentials modulo a natural gauge invariance for the vector field term.

KEYWORDS
Inverse problems, relativistic Schrödinger equation, Carleman estimates, partial boundary data

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain with C^2 boundary. For $T > \operatorname{diam}(\Omega)$, let $Q := (0,T) \times \Omega$ and denote its lateral boundary by $\Sigma := (0,T) \times \partial\Omega$. Consider the linear hyperbolic partial differential operator of second order with time-dependent coefficients:

$$\mathcal{L}_{\mathcal{A},q} u := \Big\{ (\partial_t + A_0(t,x))^2 - \sum_{j=1}^{n} (\partial_j + A_j(t,x))^2 + q(t,x) \Big\} u = 0, \ (t,x) \in Q. \quad (1)$$

We denote $\mathcal{A} = (A_0, \cdots, A_n)$ and $A = (A_1, \cdots, A_n)$. Then $\mathcal{A} = (A_0, A)$. We assume that $\mathcal{A}$ is $\mathbb{R}^{1+n}$ valued with coefficients in $C_c^\infty(Q)$ and $q \in L^\infty(Q)$. The operator (1) is known as the relativistic Schrödinger equation and appears in quantum mechanics and general relativity [20, Chap. XII]. In this paper, we study an inverse problem related to this operator. More precisely, we are interested in determining the coefficients in

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(1) from certain measurements made on suitable subsets of the topological boundary of Q.

Starting with the work of Bukhgeìm and Klibanov [6], there has been extensive work in the literature related to inverse boundary value problems for second order linear hyperbolic PDE. For the case when $\mathcal{A}$ is 0 and q is time-independent, the unique determination of q from full lateral boundary Dirichlet to Neumann data was addressed by Rakesh and Symes in [22]. Isakov in [16] considered the same problem with an additional time-independent time derivative perturbation, that is, with $\mathcal{A} = (A_0(x), 0)$ and $q(x)$ and proved uniqueness results. The results in [22] and [16] were proved using geometric optics solutions inspired by the work of Sylvester and Uhlmann [30]. For the case of time-independent coefficients, another powerful tool to prove uniqueness results is the boundary control (BC) method pioneered by Belishev, see [3–5]. Later it was developed by Belishev, Kurylev, Katchalov, Lassas, Eskin and others; see [17] and references therein. Eskin in [10,12] developed a new approach based on the BC method for determining the time-independent vector and scalar potentials assuming $A_0 = 0$ in (1). Hyperbolic inverse problems for time-independent coefficients have been extensively studied by Yamamoto and his collaborators as well; see [1,2,8,13,14].

Inverse problems involving time-dependent first and zeroth order perturbations focusing on the cases $\mathcal{A} = 0$ or when $\mathcal{A}$ is of the form $(A_0, 0)$ have been well studied in prior works. We refer to [23,24,26,28] for some works in this direction.

Eskin in [11] considered full first and zeroth order time-dependent perturbations of the wave equation in a Riemannian manifold set-up and proved uniqueness results (for the first order term, uniqueness modulo a natural gauge invariance) from boundary Dirichlet-to-Neumann data, under the assumption that the coefficients are analytic in time. Salazar removed the analyticity assumption of Eskin in [25], and proved that the unique determination of vector and scalar potential modulo a natural gauge invariance is possible from Dirichlet-to-Neumann data on the boundary. In a recent work of Stefanov and Yang in [27],[1] they proved stability estimates for the recovery of light ray transforms of time-dependent first- and zeroth-order perturbations for the wave equation in a Riemannian manifold setting from certain local Dirichlet to Neumann map. Their results, in particular, would give uniqueness results in suitable subsets of the domain recovering the vector field term up to a natural gauge invariance and the zeroth-order potential term from this data.

For the case of time-dependent perturbations, if one is interested in global uniqueness results in a finite time domain, extra information in addition to Dirichlet-to-Neumann data is required to prove uniqueness results. Isakov in [15] proved unique-determination of time-dependent potentials (assuming $\mathcal{A} = 0$ in (1)) from the data set given by the Dirichlet-to-Neumann data as well as the solution and the time derivative of the solution on the domain at the final time. Recently Kian in [18] proved unique determination of time-dependent damping coefficient $A_0(t, x)$ (with $\mathcal{A}$ of the form, $\mathcal{A} = (A_0, 0)$) and the potential $q(t, x)$ from partial Dirichlet to Neumann data together with information of the solution at the final time.

In this article, we prove unique determination of time-dependent vector and scalar potentials $\mathcal{A}(t, x)$ and $q(t, x)$ appearing in (1) (modulo a gauge invariance for the vector potential) from partial boundary data. Our work extends the results of Salazar [25] in the sense that in [25], the uniqueness results are shown with full Dirichlet to Neumann boundary data and assuming such boundary measurements are available for all time. Additionally, it extends the recent work of Kian [18], since we consider

[1] We thank Plamen Stefanov for drawing our attention to the results of this paper.

the full time-dependent vector field perturbation, whereas Kian assumes only a time derivative perturbation.

The paper is organized as follows. In §2, we state the main result of the article. In §3, we prove the Carleman estimates required to prove the existence of geometric optics (GO) solutions, and in §4, we construct the required GO solutions. In §5, we derive the integral identity using which, we prove the main theorem in §6.

2. Statement of the main result

In this section, we state the main result of this article. We begin by stating precisely what we mean by gauge invariance.

2.1. Gauge Invariance

Definition 2.1. The vector potentials $\mathcal{A}^{(1)}, \mathcal{A}^{(2)} \in C_c^\infty(Q)$ are said to be gauge equivalent if there exists a $g(t,x) \in C^\infty(\overline{Q})$ such that $g(t,x) = e^{\Phi(t,x)}$ with $\Phi \in C_c^\infty(Q)$ and

$$(\mathcal{A}^{(2)} - \mathcal{A}^{(1)})(t,x) = -\frac{\nabla_{t,x} g(t,x)}{g(t,x)} = -\nabla_{t,x}\Phi(t,x).$$

Now we state the following proposition proof of which is given in [25].

Proposition 2.2. *[25] Suppose $u(t,x)$ is a solution to the following IBVP*

$$\begin{aligned} \Big[(\partial_t + A_0^{(1)}(t,x))^2 - \sum_{j=1}^n (\partial_j + A_j^{(1)}(t,x))^2 + q_1(t,x)\Big] u(t,x) &= 0 \quad \text{in } Q \\ u(0,x) = \partial_t u(0,x) &= 0 \quad \text{in } \Omega \\ u|_\Sigma &= f \end{aligned} \tag{2}$$

and $g(t,x)$ is as defined above, then $v(t,x) = g(t,x)u(t,x)$ satisfies the following IBVP

$$\begin{aligned} \Big[(\partial_t + A_0^{(2)}(t,x))^2 - \sum_{j=1}^n (\partial_j + A_j^{(2)}(t,x))^2 + q_1(t,x)\Big] v(t,x) &= 0 \quad \text{in } Q \\ v(0,x) = \partial_t v(0,x) &= 0 \quad \text{in } \Omega \\ v|_\Sigma &= f \end{aligned} \tag{3}$$

with $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ gauge equivalent. In addition if Λ_1 and Λ_2 are Dirichlet to Neumann operator associated with (2) and (3) respectively, then

$$\Lambda_1 = \Lambda_2.$$

2.2. Statement of the main result

We introduce some notation. Following [7], fix an $\omega_0 \in \mathbb{S}^{n-1}$, and define the ω_0-shadowed and ω_0-illuminated faces by

$$\partial\Omega_{+,\omega_0} := \{x \in \partial\Omega : \ \nu(x) \cdot \omega_0 \geq 0\}, \quad \partial\Omega_{-,\omega_0} := \{x \in \partial\Omega : \ \nu(x) \cdot \omega_0 \leq 0\}$$

of $\partial\Omega$, where $\nu(x)$ is outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Corresponding to $\partial\Omega_{\pm,\omega_0}$, we denote the lateral boundary parts by $\Sigma_{\pm,\omega_0} := (0,T) \times \partial\Omega_{\pm,\omega_0}$. We denote by $F = (0,T) \times F'$ and $G = (0,T) \times G'$ where F' and G' are small enough open neighbourhoods of $\partial\Omega_{+,\omega_0}$ and $\partial\Omega_{-,\omega_0}$ respectively in $\partial\Omega$.

Consider the IBVP

$$\begin{cases} \mathcal{L}_{\mathcal{A},q} u(t,x) = 0; \ (t,x) \in Q \\ u(0,x) = \phi(x), \ \partial_t u(0,x) = \psi(x); \ x \in \Omega \\ u(t,x) = f(t,x), \ (t,x) \in \Sigma. \end{cases} \tag{4}$$

For $\phi \in H^1(\Omega), \psi \in L^2(\Omega)$ and $f \in H^1(\Sigma)$, (4) has a unique solution $u \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^1(\Omega))$ and furthermore $\partial_\nu u \in L^2(\Sigma)$; see [17,19]. Thus we have $u \in H^1(Q)$. Therefore we can define our input-output operator $\Lambda_{\mathcal{A},q}$ by

$$\Lambda_{\mathcal{A},q}(\phi, \psi, f) = (\partial_\nu u|_G, u|_{t=T}) \tag{5}$$

where u is the solution to (4). The operator

$$\Lambda_{\mathcal{A},q} : H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma) \ \to \ H^1(\Omega) \times L^2(G)$$

is a continuous linear map which follows from the well-posedness of the IBVP given by Equation (4) (see [17,19]). A natural question is whether this input-output operator uniquely determines the time-dependent perturbations $\mathcal{A}$ and q. We now state our main result.

Theorem 2.3. *Let $\left(\mathcal{A}^{(1)}, q_1\right)$ and $\left(\mathcal{A}^{(2)}, q_2\right)$ be two sets of vector and scalar potentials such that each $A_j^{(i)} \in C_c^\infty(Q)$ and $q_i \in L^\infty(Q)$ for $i = 1,2$ and $0 \leq j \leq n$. Let u_i be solutions to (4) when $(\mathcal{A}, q) = (\mathcal{A}^{(i)}, q_i)$ and $\Lambda_{\mathcal{A}^{(i)},q_i}$ for $i = 1,2$ be the input-output operators defined by (5) corresponding to u_i. If*

$$\Lambda_{\mathcal{A}^{(1)},q_1}(\phi, \psi, f) = \Lambda_{\mathcal{A}^{(2)},q_2}(\phi, \psi, f), \ \textit{for all } (\phi, \psi, f) \in H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma),$$

then there exists a function $\Phi \in C_c^\infty(Q)$ such that

$$(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})(t,x) = \nabla_{t,x}\Phi(t,x); \ \textit{and } q_1(t,x) = q_2(t,x), \ (t,x) \in Q.$$

Remark 1. We have stated the above result for vector potentials in $C_c^\infty(Q)$ for simplicity. We can, in fact, prove for the case in which the vector potentials $\mathcal{A} \in W^{1,\infty}(Q)$ provided they are identical on the boundary ∂Q, see [18].

3. Carleman Estimate

We denote by $H^1_{\mathrm{scl}}(Q)$, the semiclassical Sobolev space of order 1 on Q with the following norm

$$||u||_{H^1_{\mathrm{scl}}(Q)} = ||u||_{L^2(Q)} + ||h\nabla_{t,x}u||_{L^2(Q)},$$

and for $Q = \mathbb{R}^{1+n}$ we denote by $H^s_{\mathrm{scl}}(\mathbb{R}^{1+n})$, the Sobolev space of order s with the norm given by

$$||u||^2_{H^s_{\mathrm{scl}}(\mathbb{R}^{1+n})} = ||\langle hD\rangle^s u||^2_{L^2(\mathbb{R}^{1+n})} = \int\limits_{\mathbb{R}^{1+n}} (1 + h^2\tau^2 + h^2|\xi|^2)^s \, |\widehat{u}(\tau,\xi)|^2 \, \mathrm{d}\tau\mathrm{d}\xi.$$

In this section, we derive a Carleman estimate involving boundary terms for (1) conjugated with a linear weight. We use this estimate to control boundary terms over subsets of the boundary where measurements are not available. Our proof follows from modifications of the Carleman estimate given in [18]. Since we work in a semiclassical setting, we prefer to give the proof for the sake of completeness.

Theorem 3.1. *Let $\varphi(t,x) := t + x\cdot\omega$, where $\omega \in \mathbb{S}^{n-1}$ is fixed. Assume that $A_j \in C^\infty_c(Q)$ for $0 \le j \le n$ and $q \in L^\infty(Q)$. Then the Carleman estimate*

$$\begin{aligned}
&h\left(e^{-\varphi/h}\partial_\nu\varphi\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{+,\omega})} + h\left(e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot), e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot)\right)_{L^2(\Omega)} \\
&+ \|e^{-\varphi/h}u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\partial_t u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\nabla_x u\|^2_{L^2(Q)} \\
&\le C\Big(\|he^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}u\|^2_{L^2(Q)} + \left(e^{-\varphi(T,\cdot)/h}u(T,\cdot), e^{-\varphi(T,\cdot)/h}u(T,\cdot)\right)_{L^2(\Omega)} \\
&+ h\left(e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot), e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot)\right)_{L^2(\Omega)} + h\left(e^{-\varphi/h}\left(-\partial_\nu\varphi\right)\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{-,\omega})}\Big)
\end{aligned} \tag{6}$$

holds for all $u \in C^2(Q)$ with

$$u|_\Sigma = 0, \ u|_{t=0} = \partial_t u|_{t=0} = 0,$$

and h small enough.

Proof. To prove the estimate (6), we will use a convexification argument used in [18]. Consider the following perturbed weight function

$$\widetilde{\varphi}(t,x) = \varphi(t,x) - \frac{ht^2}{2\varepsilon}. \tag{7}$$

We first consider the conjugated operator

$$\Box_{\varphi,\varepsilon} := h^2 e^{-\widetilde{\varphi}/h}\Box e^{\widetilde{\varphi}/h}. \tag{8}$$

For $v \in C^2(Q)$ satisfying $v|_\Sigma = v|_{t=0} = \partial_t v|_{t=0} = 0$, consider the L^2 norm of $\Box_{\varphi,\varepsilon}$:

$$\int_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t.$$

Expanding (8), we get,

$$\Box_{\varphi,\varepsilon} v(t,x) = \left(h^2\Box + h\Box\widetilde{\varphi} + \left(|\partial_t\widetilde{\varphi}|^2 - |\nabla_x\widetilde{\varphi}|^2\right) + 2h\left(\partial_t\widetilde{\varphi}\partial_t - \nabla_x\widetilde{\varphi}\cdot\nabla_x\right)\right)v(t,x).$$

We write this as

$$\Box_{\varphi,\varepsilon} v(t,x) = P_1 v(t,x) + P_2 v(t,x),$$

where

$$\begin{aligned} P_1 v(t,x) &= \left(h^2\Box + h\Box\widetilde{\varphi} + \left(|\partial_t\widetilde{\varphi}|^2 - |\nabla_x\widetilde{\varphi}|^2\right)\right)v(t,x) \\ &= \left(h^2\Box + \frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)v(t,x), \end{aligned}$$

and

$$\begin{aligned} P_2 v(t,x) &= 2h\left(\partial_t\widetilde{\varphi}\partial_t - \nabla_x\widetilde{\varphi}\cdot\nabla_x\right)v(t,x) \\ &= 2h\left(\left(1 - \frac{ht}{\varepsilon}\right)\partial_t - \omega\cdot\nabla_x\right)v(t,x). \end{aligned}$$

Now

$$\begin{aligned} \int_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t &\geq 2\int_Q \mathrm{Re}\left(P_1 v(t,x)\overline{P_2 v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &= 4h^3\int_Q \mathrm{Re}\left(\Box v(t,x)\left(1 - \frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &\quad - 4h^3\int_Q \mathrm{Re}\left(\Box v(t,x)\omega\cdot\overline{\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &\quad + 4h\int_Q \mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)v(t,x)\left(1 - \frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &\quad - 4h\int_Q \mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)v(t,x)\omega\cdot\overline{\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We first simplify I_1. We have

$$\begin{aligned}
I_1 &= 4h^3 \int_Q \mathrm{Re}\left(\Box v(t,x)\left(1-\frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= 2h^3 \int_Q \frac{\partial}{\partial t}|\partial_t v(t,x)|^2\left(1-\frac{ht}{\varepsilon}\right)\mathrm{d}x\mathrm{d}t - 4h^3\int_Q \mathrm{Re}\left(\Delta v(t,x)\left(1-\frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= 2h^3\left(1-\frac{hT}{\varepsilon}\right)\int_\Omega \left(|\partial_t v(T,x)|^2+|\nabla_x v(T,x)|^2\right)\mathrm{d}x + \frac{2h^4}{\varepsilon}\int_Q \left(|\partial_t v(t,x)|^2+|\nabla_x v(t,x)|^2\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

In the above steps, we used integration by parts combined with the hypotheses that $v|_\Sigma = v|_{t=0} = \partial_t v|_{t=0} = 0$. Note that $v|_\Sigma = 0$ would imply that $\partial_t v = 0$ on Σ.

Now we consider I_2. We have

$$I_2 = -4h^3\int_Q \mathrm{Re}\left(\Box v(t,x)\omega\cdot\overline{\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t.$$

We have

$$\begin{aligned}
I_2 &= -4h^3\mathrm{Re}\int_Q \partial_t^2 v(t,x)\overline{\omega\cdot\nabla_x v(t,x)}\mathrm{d}x\mathrm{d}t + 4h^3\mathrm{Re}\int_Q \Delta v(t,x)\overline{\omega\cdot\nabla_x v(t,x)}\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int_Q \partial_t\left(\partial_t v(t,x)\overline{\omega\cdot\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t + 4h^3\mathrm{Re}\int_Q \partial_t v(t,x)\overline{\omega\cdot\nabla_x\partial_t v(t,x)}\mathrm{d}x\mathrm{d}t \\
&\quad + 4h^3\mathrm{Re}\int_Q \nabla_x\cdot\left(\nabla_x v(t,x)\overline{\omega\cdot\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t - 4h^3\mathrm{Re}\int_Q \nabla_x v(t,x)\cdot\nabla_x\left(\overline{\omega\cdot\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int_\Omega \partial_t v(T,x)\overline{\omega\cdot\nabla_x v(T,x)}\mathrm{d}x + 2h^3\int_Q \nabla_x\cdot\left(\omega|\partial_t v(t,x)|^2\right)\mathrm{d}x\mathrm{d}t \\
&\quad + 2h^3\mathrm{Re}\int_\Sigma \partial_\nu v(t,x)\overline{\omega\cdot\nabla_x v(t,x)}\mathrm{d}S_x\mathrm{d}t - 2h^3\int_Q \nabla_x\cdot\left(\omega|\nabla_x v|^2\right)\mathrm{d}x\mathrm{d}t \\
&= -4h^3\mathrm{Re}\int_\Omega \partial_t v(T,x)\overline{\omega\cdot\nabla_x v(T,x)}\mathrm{d}x + 2h^3\int_\Sigma \omega\cdot\nu|\partial_t v(t,x)|^2\mathrm{d}S_x\mathrm{d}t + 2h^3\int_\Sigma \omega\cdot\nu|\partial_\nu v|^2\mathrm{d}S_x\mathrm{d}t.
\end{aligned}$$

In deriving the above equation, we used the fact that

$$2h^3\mathrm{Re}\int_\Sigma \partial_\nu v(t,x)\overline{\omega\cdot\nabla_x v(t,x)}\mathrm{d}S_x\mathrm{d}t = 2h^3\int_\Sigma \omega\cdot\nu|\partial_\nu v|^2\mathrm{d}S_x\mathrm{d}t,$$

since $v = 0$ on Σ. Also note that $|\nabla_x v| = |\partial_\nu v|$.

Next we consider I_3. We have

$$
\begin{aligned}
I_3 &= 4h \int_Q \mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right) v(t,x)\left(1 - \frac{ht}{\varepsilon}\right)\overline{\partial_t v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= 2h \int_Q \left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)\left(1 - \frac{ht}{\varepsilon}\right)\partial_t |v(t,x)|^2 \mathrm{d}x\mathrm{d}t \\
&= 2\int_\Omega \left(\frac{h^3T^2}{\varepsilon^2} - \frac{2h^2T}{\varepsilon} - \frac{h^3}{\varepsilon}\right)\left(1 - \frac{hT}{\varepsilon}\right)|v(T,x)|^2\mathrm{d}x \\
&\qquad - 2\int_Q \left[\left(\frac{2h^3t}{\varepsilon^2} - \frac{2h^2}{\varepsilon}\right)\left(1 - \frac{ht}{\varepsilon}\right) - \frac{h^2}{\varepsilon}\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)\right]|v(t,x)|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Finally, we consider I_4. This is

$$
\begin{aligned}
I_4 &:= -4h \int_Q \mathrm{Re}\left(\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right) v(t,x)\omega \cdot \overline{\nabla_x v(t,x)}\right)\mathrm{d}x\mathrm{d}t \\
&= -4h \int_Q \left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)\nabla \cdot \left(\omega |v|^2\right)\mathrm{d}x\mathrm{d}t \\
&= 0 \quad \text{since } v = 0 \text{ on } \Sigma.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t &\geq 2h^3\left(1 - \frac{hT}{\varepsilon}\right)\int_\Omega \left(|\partial_t v(T,x)|^2 + |\nabla_x v(T,x)|^2\right)\mathrm{d}x \\
&\quad + \frac{2h^4}{\varepsilon}\int_Q \left(|\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2\right)\mathrm{d}x\mathrm{d}t \\
&\quad - 4h^3 \mathrm{Re}\int_\Omega \partial_t v(T,x)\overline{\omega \cdot \nabla_x v(T,x)}\mathrm{d}x + 2h^3 \int_\Sigma \omega \cdot \nu |\partial_t v(t,x)|^2 \mathrm{d}S_x\mathrm{d}t \\
&\quad + 2h^3 \int_\Sigma \omega \cdot \nu |\partial_\nu v|^2 \mathrm{d}S_x\mathrm{d}t + 2\int_\Omega \left(\frac{h^3T^2}{\varepsilon^2} - \frac{2h^2T}{\varepsilon} - \frac{h^3}{\varepsilon}\right)\left(1 - \frac{hT}{\varepsilon}\right)|v(T,x)|^2\mathrm{d}x \\
&\qquad - 2\int_Q \left[\left(\frac{2h^3t}{\varepsilon^2} - \frac{2h^2}{\varepsilon}\right)\left(1 - \frac{ht}{\varepsilon}\right) - \frac{h^2}{\varepsilon}\left(\frac{h^2t^2}{\varepsilon^2} - \frac{2ht}{\varepsilon} - \frac{h^2}{\varepsilon}\right)\right]|v(t,x)|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Choosing ε and h small enough, we have

$$\int_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t \geq \frac{2h^4}{\varepsilon}\left(\int_Q |\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t\right) + ch^3 \int_\Omega |\partial_t v(T,x)|^2 \mathrm{d}x$$
$$+ 2h^3 \int_\Sigma \omega\cdot\nu(x)|\partial_\nu v(t,x)|^2 \mathrm{d}S_x\mathrm{d}t - ch^3 \int_\Omega |\nabla_x v(T,x)|^2 \mathrm{d}x$$
$$- ch^2 \int_\Omega |v(T,x)|^2 \mathrm{d}x + \frac{ch^2}{\varepsilon}\int_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t. \tag{9}$$

Now we consider the conjugated operator $\mathcal{L}_{\varphi,\varepsilon} := h^2 e^{-\frac{\widetilde{\varphi}}{h}} \mathcal{L}_{\mathcal{A},q} e^{\frac{\widetilde{\varphi}}{h}}$. We have

$$\mathcal{L}_{\varphi,\varepsilon} v(t,x) = h^2\left(e^{-\widetilde{\varphi}/h}\left(\Box + 2A_0\partial_t - 2A\cdot\nabla_x + \widetilde{q}\right) e^{\widetilde{\varphi}/h} v(t,x)\right),$$

where

$$\widetilde{q} = q + |A_0|^2 - |A|^2 + \partial_t A_0 - \nabla_x \cdot A.$$

We write

$$\mathcal{L}_{\varphi,\varepsilon} v(t,x) = \Box_{\varphi,\varepsilon} v(t,x) + \widetilde{P} v(t,x),$$

where

$$\widetilde{P} v(t,x) = h^2\left(e^{-\widetilde{\varphi}/h}\left(2A_0\partial_t - 2A\cdot\nabla_x + \widetilde{q}\right) e^{\widetilde{\varphi}/h} v(t,x)\right). \tag{10}$$

By triangle inequality,

$$\int_Q |\mathcal{L}_{\varphi,\varepsilon} v(t,x)|^2 \,\mathrm{d}x\mathrm{d}t \geq \frac{1}{2}\int_Q |\Box_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t - \int_Q |\widetilde{P} v(t,x)|^2 \mathrm{d}x\mathrm{d}t. \tag{11}$$

We have

$$\int_Q \left|\widetilde{P} v(t,x)\right|^2 \mathrm{d}x\mathrm{d}t = \int_Q \left|\left[h^2\left(2A_0\partial_t - 2A\cdot\nabla_x + \widetilde{q}\right) v + h\left\{2A_0\left(1 - \frac{ht}{\varepsilon}\right) - 2\omega\cdot A\right\} v\right]\right|^2 \mathrm{d}x\mathrm{d}t$$
$$\leq Ch^4\left(\|A_0\|^2_{L^\infty(Q)} \int_Q |\partial_t v(t,x)|^2 \mathrm{d}x\mathrm{d}t + \|A\|^2_{L^\infty(Q)} \int_Q |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t\right)$$
$$+ Ch^2\left(\left(h^2\|\widetilde{q}\|^2_{L^\infty(Q)} + \|A\|^2_{L^\infty(Q)}\right)\int_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t + \|A_0\|^2_{L^\infty(Q)} \int_Q \left(1 - \frac{ht}{\varepsilon}\right)^2 |v(t,x)|^2 \mathrm{d}x\mathrm{d}t\right).$$

Choosing h small enough, we have,

$$\int_Q \left|\widetilde{P}v(t,x)\right|^2 \mathrm{d}x\mathrm{d}t \leq Ch^4 \left(\|A_0\|^2_{L^\infty(Q)} \int_Q |\partial_t v(t,x)|^2 \mathrm{d}x\mathrm{d}t + \|A\|^2_{L^\infty(Q)} \int_Q |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t \right) + Ch^2 \left(\|A_0\|^2_{L^\infty(Q)} + \|A\|^2_{L^\infty(Q)} \right) \int_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t + Ch^4 \|\widetilde{q}\|^2_{L^\infty(Q)} \int_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t. \tag{12}$$

Using (9) and (12) in (11) and taking ε small enough, we have that there exists a $C > 0$ depending only on ε, T, Ω, $\mathcal{A}$ and q such that

$$\int_Q |\mathcal{L}_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t \geq Ch^2 \Bigg\{ h^2 \left(\int_Q |\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2 \mathrm{d}x\mathrm{d}t \right) + \int_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t - \int_\Omega |v(T,x)|^2 \mathrm{d}x \Bigg\} + Ch^3 \int_\Omega |\partial_t v(T,x)|^2 \mathrm{d}x + h^3 \int_\Sigma \omega \cdot \nu(x) |\partial_\nu v(t,x)|^2 \mathrm{d}S_x \mathrm{d}t - Ch^3 \int_\Omega |\nabla_x v(T,x)|^2 \mathrm{d}x,$$

and this inequality holds for h small enough. After dividing by h^2, we get

$$C \left(\int_Q \left(|h\partial_t v(t,x)|^2 + |h\nabla_x v(t,x)|^2 \right) \mathrm{d}x\mathrm{d}t + \int_Q |v(t,x)|^2 \mathrm{d}x\mathrm{d}t \right) + Ch \int_\Omega |\partial_t v(T,x)|^2 \mathrm{d}x + h \int_\Sigma \omega \cdot \nu(x) |\partial_\nu v(t,x)|^2 \mathrm{d}S_x \mathrm{d}t \leq \frac{1}{h^2} \int_Q |\mathcal{L}_{\varphi,\varepsilon} v(t,x)|^2 \mathrm{d}x\mathrm{d}t + Ch \int_\Omega |\nabla_x v(T,x)|^2 \mathrm{d}x + C \int_\Omega |v(T,x)|^2 \mathrm{d}x. \tag{13}$$

Let us now substitute $v(t,x) = e^{-\frac{\bar{\varphi}}{h}} u(t,x)$. We have

$$\begin{aligned} &he^{-\varphi/h} \partial_t u(t,x) = he^{-t^2/2\varepsilon} \partial_t v + e^{-\varphi/h} \left(1 - \frac{ht}{\varepsilon} \right) u, \\ &he^{-\varphi/h} \nabla_x u = he^{-t^2/2\varepsilon} \nabla_x v + e^{-\varphi/h} \omega u, \\ &\partial_\nu v(t,x)|_\Sigma = e^{-\varphi/h} \partial_\nu u|_\Sigma, \quad \text{since } u = 0 \text{ on } \Sigma. \end{aligned}$$

Using the above equalities and the triangle inequality, we then have for h small enough,

$$
\begin{aligned}
&h\int_{\Omega} e^{-2\varphi/h}|\partial_t u(T,x)|^2\mathrm{d}x + h\int_{\Sigma_{+,\omega}} e^{-2\varphi/h}|\partial_\nu u(t,x)|^2|\omega\cdot\nu(x)|\mathrm{d}S_x\mathrm{d}t\\
&+h^2\int_{Q} e^{-2\varphi/h}\left(|\partial_t u(t,x)|^2+|\nabla_x u(t,x)|^2\right)\mathrm{d}x\mathrm{d}t + \int_{Q} e^{-2\varphi/h}|u(t,x)|^2\mathrm{d}x\mathrm{d}t\\
&\leq C\Bigg(h^2\int_{Q} e^{-2\varphi/h}|\mathcal{L}_{\mathcal{A},q}u(t,x)|^2\mathrm{d}x\mathrm{d}t + h\int_{\Omega} e^{-2\varphi/h}|\nabla_x u(T,x)|^2\mathrm{d}x\\
&+\int_{\Omega} e^{-2\varphi/h}|u(T,x)|^2\mathrm{d}x + h\int_{\Sigma_{-,\omega}} e^{-2\varphi/h}|\partial_\nu u(t,x)|^2|\omega\cdot\nu(x)|\mathrm{d}S_x\mathrm{d}t\Bigg).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&h\left(e^{-\varphi/h}\partial_\nu\varphi\partial_\nu u, e^{-\phi/h}\partial_\nu u\right)_{L^2(\Sigma_{+,\omega})} + h\left(e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot), e^{-\varphi(T,\cdot)/h}\partial_t u(T,\cdot)\right)_{L^2(\Omega)}\\
&+\|e^{-\varphi/h}u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\partial_t u\|^2_{L^2(Q)} + \|he^{-\varphi/h}\nabla_x u\|^2_{L^2(Q)}\\
&\leq C\Bigg(\|he^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}u\|^2_{L^2(Q)} + \left(e^{-\varphi(T,\cdot)/h}u(T,\cdot), e^{-\varphi(T,\cdot)/h}u(T,\cdot)\right)_{L^2(\Omega)}\\
&+h\left(e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot), e^{-\varphi(T,\cdot)/h}\nabla_x u(T,\cdot)\right)_{L^2(\Omega)} + h\left(e^{-\varphi/h}\left(-\partial_\nu\varphi\right)\partial_\nu u, e^{-\varphi/h}\partial_\nu u\right)_{L^2(\Sigma_{-,\omega})}\Bigg).
\end{aligned}
$$

This completes the proof. □

In particular, it follows from the previous calculations that for $u\in C_c^\infty(Q)$,

$$
\|u\|_{H^1_{\mathrm{scl}}(Q)} \leq \frac{C}{h}\|\mathcal{L}_\varphi u\|_{L^2(Q)}. \tag{14}
$$

To show the existence of suitable solutions to (1), we need to shift the Sobolev index by -1 in (14). This we do in the next lemma.

Lemma 3.2. *Let $\varphi(t,x)=t+x\cdot\omega$ and $\mathcal{L}_\varphi := h^2e^{-\varphi/h}\mathcal{L}_{\mathcal{A},q}e^{\varphi/h}$. There exists an $h_0>0$ such that*

$$
\|v\|_{L^2(\mathbb{R}^{1+n})} \leq \frac{C}{h}\|\mathcal{L}_\varphi v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}, \tag{15}
$$

and

$$
\|v\|_{L^2(\mathbb{R}^{1+n})} \leq \frac{C}{h}\|\mathcal{L}^*_\varphi v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \tag{16}
$$

for all $v\in C_c^\infty(Q)$, $0<h\leq h_0$.

Proof. We give the proof of the estimate in (15) and that of (16) follows similarly.

We follow arguments used in [9]. We again consider the convexified weight

$$\widetilde{\varphi}(t,x) = t + x\cdot\omega - \frac{ht^2}{2\varepsilon},$$

and as before consider the convexified operator:

$$\Box_{\varphi,\varepsilon} := h^2 e^{-\widetilde{\varphi}/h}\Box e^{\widetilde{\varphi}/h}.$$

From the properties of pseudodifferential operators, we have

$$\langle hD\rangle^{-1}\left(\Box_{\varphi,\varepsilon}\right)\langle hD\rangle = \Box_{\varphi,\varepsilon} + hR_1$$

where R_1 is a semi-classical pseudo-differential operator of order 1. Now

$$\|\Box_{\varphi,\varepsilon}\langle hD\rangle v\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} = \|\langle hD\rangle^{-1}\Box_{\varphi,\varepsilon}\langle hD\rangle v\|_{L^2(\mathbb{R}^{1+n})}.$$

and by the commutator property above, we get

$$\|\Box_{\varphi,\varepsilon}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} = \|\left(\Box_{\varphi,\varepsilon} + hR_1\right)v\|^2_{L^2(\mathbb{R}^{1+n})} \geq \frac{1}{2}\|\Box_{\varphi,\varepsilon}v\|^2_{L^2(\mathbb{R}^{1+n})} - \|hR_1 v\|^2_{L^2(\mathbb{R}^{1+n})}.$$

Let $Q \subset\subset \widetilde{Q}$, and for $v \in C_c^\infty(\widetilde{Q})$, using the estimate in (9) for C_c^∞ functions combined with estimates for pseudodifferential operators, we have,

$$\|\Box_{\varphi,\varepsilon}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq \frac{Ch^2}{\varepsilon}\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} - h^2\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})}. \tag{17}$$

Using the expression for $\widetilde{P}$ (see (10)), we get, for $v \in C_c^\infty(\widetilde{Q})$ and for h small enough,

$$\|\widetilde{P}v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq Ch^2\|v\|^2_{L^2(\mathbb{R}^{1+n})},$$

and therefore

$$\|\widetilde{P}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq Ch^2\|\langle hD\rangle v\|^2_{L^2(\mathbb{R}^{1+n})} = Ch^2\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

Combining this with the estimate in (17) together with triangle inequality, we get,

$$\|\mathcal{L}_{\varphi,\varepsilon}\langle hD\rangle v\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq \frac{Ch^2}{\varepsilon}\|v\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \tag{18}$$

for all $v \in C_c^\infty(\widetilde{Q})$.

Now to complete the proof, for any $u \in C_c^\infty(Q)$, consider $v = \chi\langle hD\rangle^{-1}u$, where $\chi \in C_c^\infty(\widetilde{Q})$ with $\chi \equiv 1$ on Q. Then from (18), we have

$$\frac{Ch^2}{\varepsilon}\|\chi\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq \|\mathcal{L}_{\varphi,\varepsilon}\langle hD\rangle\chi\langle hD\rangle^{-1}u\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

The operator $\langle hD\rangle\chi\langle hD\rangle^{-1}$ is a semiclassical pseudodifferential operator of order 0, and therefore we have

$$\mathcal{L}_{\varphi,\varepsilon}\langle hD\rangle\chi\langle hD\rangle^{-1}u = \langle hD\rangle\chi\langle hD\rangle^{-1}\mathcal{L}_{\varphi,\varepsilon} + hR_1,$$

where R_1 is a semiclassical pseudodifferential operator of order 1.

$$\frac{Ch^2}{\varepsilon}\|\chi\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq \|\mathcal{L}_{\varphi,\varepsilon}u\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} + h^2\|u\|^2_{L^2(\mathbb{R}^{1+n})}.$$

Finally, write

$$\langle hD\rangle^{-1}u = \chi\langle hD\rangle^{-1}u + (1-\chi)\langle hD\rangle^{-1}u,$$

where χ is as above. Then

$$\|\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq \frac{1}{2}\|\chi\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} - \|(1-\chi)\langle hD\rangle^{-1}u\|^2_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

Since $(1-\chi)\langle hD\rangle^{-1}$ is a smoothing semiclassical pseudodifferential operator, taking h small enough, and arguing as in the proof of the Carleman estimate, we get,

$$\|\mathcal{L}_\varphi u\|^2_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})} \geq Ch^2\|u\|^2_{L^2(\mathbb{R}^{1+n})}.$$

Cancelling out the h^2 term, we finally have,

$$\|u\|_{L^2(\mathbb{R}^{1+n})} \leq \frac{C}{h}\|\mathcal{L}_\varphi u\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}.$$

This completes the proof. □

Proposition 3.3. *Let φ, $\mathcal{A}$ and q be as in Theorem 3.1. For $h > 0$ small enough and $v \in L^2(Q)$, there exists a solution $u \in H^1(Q)$ of*

$$\mathcal{L}_\varphi u = v,$$

satisfying the estimate

$$\|u\|_{H^1_{\mathrm{scl}}(Q)} \leq \frac{C}{h}\|v\|_{L^2(\mathbb{R}^{1+n})},$$

where $C > 0$ is a constant independent of h.

Proof. The proof uses standard functional analysis arguments. Consider the space $S := \left\{\mathcal{L}^*_\varphi u : u \in C^\infty_c(Q)\right\}$ as a subspace of $H^{-1}(\mathbb{R}^{1+n})$ and define a linear form L on S by

$$L(\mathcal{L}^*_\varphi z) = \int_Q z(t,x)v(t,x)\mathrm{d}x\mathrm{d}t, \text{ for } z \in C^\infty_c(Q).$$

This is a well-defined continuous linear functional by the Carleman estimate (16) . We have

$$\left|L(\mathcal{L}_\varphi^* z)\right| \leq \|z\|_{L^2(Q)}\|v\|_{L^2(Q)} \leq \frac{C}{h}\|v\|_{L^2(Q)}\|\mathcal{L}_\varphi^* z\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^{1+n})}, z \in C_c^\infty(Q).$$

By Hahn-Banach theorem, we can can extend L to $H^{-1}(\mathbb{R}^{1+n})$ (still denoted as L) and it satisfies $\|L\| \leq \frac{C}{h}\,||v||_{L^2(Q)}$. By Riesz representation theorem, there exists a unique $u \in H^1(\mathbb{R}^{1+n})$ such that

$$L(f) = \langle f, u\rangle_{L^2(\mathbb{R}^{1+n})} \text{ for all } f \in H^{-1}(\mathbb{R}^{1+n}) \text{ with } \|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^{1+n})} \leq \frac{C}{h}\|v\|_{L^2(Q)}.$$

Taking $f = \mathcal{L}_\varphi^* z$, for $z \in C_c^\infty(Q)$, we get

$$L(\mathcal{L}_\varphi^* z) = \langle \mathcal{L}_\varphi^* z, u\rangle_{L^2(\mathbb{R}^{1+n})} = \langle z, \mathcal{L}_\varphi u\rangle_{L^2(\mathbb{R}^{1+n})}.$$

Therefore for all $z \in C_c^\infty(Q)$,

$$\langle z, \mathcal{L}_\varphi u\rangle = \langle z, v\rangle.$$

Hence

$$\mathcal{L}_\varphi u = v \text{ in } Q \text{ with } \|u\|_{H^1_{\mathrm{scl}}(Q)} \leq \frac{C}{h}\|v\|_{L^2(Q)}.$$

This completes the proof of the proposition. □

4. Construction of geometric optics solutions

In this section we construct geometric optics solutions for $\mathcal{L}_{\mathcal{A},q}u = 0$ and its adjoint operator $\mathcal{L}^*_{\mathcal{A},q}u = \mathcal{L}_{-\mathcal{A},\bar{q}}u = 0$.

Proposition 4.1. *Let $\mathcal{L}_{\mathcal{A},q}$ be as in* (1).

(1) (Exponentially decaying solutions) There exists an $h_0 > 0$ such that for all $0 < h \leq h_0$, we can find $v \in H^1(Q)$ satisfying $\mathcal{L}_{-\mathcal{A},\bar{q}}v = 0$ of the form

$$v_d(t,x) = e^{-\frac{\varphi}{h}}\left(B_d(t,x) + hR_d(t,x;h)\right), \tag{19}$$

where $\varphi(t,x) = t + x\cdot\omega$,

$$B_d(t,x) = \exp\left(-\int_0^\infty (1,-\omega)\cdot\mathcal{A}(t+s,x-s\omega)\mathrm{d}s\right) \tag{20}$$

with $\zeta \in (1,-\omega)^\perp$ and $R_d \in H^1(Q)$ satisfies

$$\|R_d\|_{H^1_{\mathrm{scl}}(Q)} \leq C. \tag{21}$$

(2) (Exponentially growing solutions) There exists an $h_0 > 0$ such that for all $0 < h \leq h_0$, we can find $v \in H^1(Q)$ satisfying $\mathcal{L}_{\mathcal{A},q} v = 0$ of the form

$$v_g(t,x) = e^{\frac{\varphi}{h}} \left(B_g(t,x) + hR_g(t,x;h) \right), \tag{22}$$

where $\varphi(t,x) = t + x \cdot \omega$,

$$B_g(t,x) = e^{-i\zeta\cdot(t,x)} \exp\left(\int_0^\infty (1,-\omega) \cdot \mathcal{A}(t+s, x-s\omega) \mathrm{d}s \right) \tag{23}$$

with $\zeta \in (1,-\omega)^\perp$ and $R_g \in H^1(Q)$ satisfies

$$\|R_g\|_{H^1_{\mathrm{scl}}(Q)} \leq C. \tag{24}$$

Proof. We have

$$\mathcal{L}_{A,q} v = \Box v + 2A_0 \partial_t v - 2A \cdot \nabla_x v + \left(\partial_t A_0 - \nabla_x \cdot A + |A_0|^2 - |A|^2 + q \right) v.$$

Letting v of the form

$$v(t,x) = e^{\frac{\varphi}{h}} \left(B_g + hR_g \right),$$

and setting the term involving h^{-1} to be 0, we get,

$$(1,-\omega) \cdot \left(\nabla_{t,x} B_g + (A_0, A) B_g \right) = 0.$$

One solution of this equation is

$$B_g(t,x) = \exp\left(\int_0^\infty (1,-\omega) \cdot \mathcal{A}(t+s, x-s\omega) \mathrm{d}s \right).$$

Alternately, another solution is

$$B_g(t,x) = e^{-\mathrm{i}\zeta\cdot(t,x)} \exp\left(\int_0^\infty (1,-\omega) \cdot \mathcal{A}(t+s, x-s\omega) \mathrm{d}s \right),$$

provided $\zeta \in (1,-\omega)^\perp$.

Now we have

$$\mathcal{L}^*_{\mathcal{A},q} = \mathcal{L}_{-\mathcal{A},\overline{q}}.$$

For this adjoint operator, the equation satisfied by B_d is

$$(1,-\omega) \cdot \left(\nabla_{t,x} B_g - (A_0, A) B_g \right) = 0.$$

We let

$$B_d(t,x) = \exp\left(-\int_0^\infty (1,-\omega)\cdot \mathcal{A}(t+s,x-s\omega)\mathrm{d}s\right).$$

Now R_g satisfies

$$\mathcal{L}_\varphi R_g = -h\mathcal{L}_{\mathcal{A},q}B_g.$$

Then using the estimate in Proposition 3.3, we get that

$$\|R_g\|_{H^1_{\mathrm{scl}}(Q)} \leq C\|\mathcal{L}_{\mathcal{A},q}B_g\|_{L^2(Q)}.$$

Similarly,

$$\|R_d\|_{H^1_{\mathrm{scl}}(Q)} \leq C\|\mathcal{L}_{-\mathcal{A},\overline{q}}B_d\|_{L^2(Q)}.$$

The proof is complete. □

5. Integral Identity

In this section, we derive an integral identity involving the coefficients $\mathcal{A}$ and q using the geometric optics solutions described in the previous section.

Let u_i be the solutions to the following initial boundary value problems with vector field coefficient $\mathcal{A}^{(i)}$ and scalar potential q_i for $i = 1, 2$.

$$\begin{cases} L_{\mathcal{A}^{(i)},q_i}u_i(t,x) = 0;\ (t,x)\in Q \\ u_i(0,x) = \phi(x),\ \partial_t u_i(0,x) = \psi(x);\ x\in\Omega \\ u_i(t,x) = f(t,x),\ (t,x)\in\Sigma. \end{cases} \tag{25}$$

Let us denote

$$\begin{aligned} &u(t,x) := (u_1-u_2)(t,x) \\ &\mathcal{A}(t,x) := \mathcal{A}^{(2)} - \mathcal{A}^{(1)}(t,x) := (A_0(t,x), A_1(t,x), \cdots, A_n(t,x)) \\ &\widetilde{q}_i := \partial_t A_0^{(i)} - \nabla_x\cdot\mathcal{A}^{(i)} + |A_0^{(i)}|^2 - |\mathcal{A}^{(i)}|^2 + q_i \\ &\widetilde{q} := \widetilde{q}_2 - \widetilde{q}_1. \end{aligned} \tag{26}$$

Then u is the solution to the following initial boundary value problem:

$$\begin{cases} L_{\mathcal{A}^{(1)},q_1}u(t,x) = -2A\cdot\nabla_x u_2 + 2A_0\partial_t u_2 + \widetilde{q}u_2 \\ u(0,x) = \partial_t u(0,x) = 0,\ x\in\Omega \\ u|_\Sigma = 0. \end{cases} \tag{27}$$

Let $v(t,x)$ of the form given by (19) be the solution to following equation

$$L_{-\mathcal{A}^{(1)},\overline{q}_1}v(t,x) = 0 \text{ in } Q. \tag{28}$$

Also let u_2 of the form given by (22) be solution to the following equation

$$L_{\mathcal{A}^{(2)},q_2}u_2(t,x)=0,\ \text{in } Q. \tag{29}$$

By the well-posedness result from [17,19], we have $u\in H^1(Q)$ and $\partial_\nu u\in L^2(\Sigma)$.

Now we multiply (27) by $\overline{v(t,x)}\in H^1(Q)$ and integrate over Q. We get, after integrating by parts, taking into account the following: $u|_\Sigma=0$, $u(T,x)=0$, $\partial_\nu u|_G=0$, $u|_{t=0}=\partial_t u|_{t=0}=0$ and $\mathcal{A}^{(1)}$ is compactly supported in Q:

$$\int\limits_Q \mathcal{L}_{\mathcal{A}^{(1)},q_1}u(t,x)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t-\int\limits_Q u(t,x)\overline{\mathcal{L}_{-\mathcal{A}^{(1)},\overline{q}_1}v(t,x)}\mathrm{d}x\mathrm{d}t=\int\limits_\Omega \partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x$$
$$-\int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t.$$

Now using the fact that $L_{-\mathcal{A}^{(1)},\overline{q}_1}v(t,x)=0$ in Q and

$$\mathcal{L}_{\mathcal{A}^{(1)},q_1}u(t,x)=-2A\cdot\nabla_x u_2+2A_0\partial_t u_2+\widetilde{q}u_2,$$

we get,

$$\int\limits_Q\left(-2A\cdot\nabla_x u_2+2A_0\partial_t u_2+\widetilde{q}u_2\right)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t=\int\limits_\Omega\partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x$$
$$-\int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t.$$

Lemma 5.1. *Let u_i for $i=1,2$ solutions to (25) with u_2 of the form (22). Let $u=u_1-u_2$, and v be of the form (19). Then*

$$h\int\limits_\Omega\partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x\to 0\ \textit{as}\ h\to 0^+. \tag{30}$$

$$h\int\limits_{\Sigma\setminus G}\partial_\nu u(t,x)\overline{v(t,x)}\mathrm{d}S_x\mathrm{d}t\to 0\ \textit{as}\ h\to 0^+. \tag{31}$$

Proof. Using (19), (21) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}
\left| h \int_\Omega \partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x \right| &\leq \int_\Omega h \left| \partial_t u(T,x) e^{-\frac{\varphi(T,x)}{h}} \overline{(B_d(T,x)+hR_d(T,x))} \right| \mathrm{d}x \\
&\leq C \left(\int_\Omega h^2 \left| \partial_t u(T,x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega \left| e^{-i\xi\cdot(T,x)} + h\overline{R_d(T,x)} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq C \left(\int_\Omega h^2 \left| \partial_t u(T,x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(1 + \|hR_d(T,\cdot)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_\Omega h^2 \left| \partial_t u(T,x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned}$$

Now using the boundary Carleman estimate (3.1), we get,

$$\begin{aligned}
h \int_\Omega \left| \partial_t u(T,x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x &\leq C \| h e^{-\varphi/h} \mathcal{L}_{\mathcal{A}^{(1)},q_1} u \|^2_{L^2(Q)} \\
&= C \| h e^{-\varphi/h} \left(2A_0 \partial_t u_2 - 2A \cdot \nabla_x u_2 + \widetilde{q} u_2 \right) \|^2_{L^2(Q)}.
\end{aligned}$$

Now substituting (22) for u_2, we get,

$$\begin{aligned}
h e^{-\varphi/h} \left(2A_0 \partial_t u_2 - 2A \cdot \nabla_x u_2 + \widetilde{q} u_2 \right) &= \left(2A_0 \partial_t \varphi - 2A \cdot \nabla_x \varphi + \widetilde{q} \right) \left(B_g + hR_g \right) \\
&\quad + h \left(2A_0 \partial_t - 2A \cdot \nabla_x + \widetilde{q} \right) \left(B_g + hR_g \right)
\end{aligned}$$

Therefore

$$\| h e^{-\varphi/h} \left(2A_0 \partial_t u_2 - 2A \cdot \nabla_x u_2 + \widetilde{q} u_2 \right) \|^2_{L^2(Q)} \leq C,$$

uniformly in h.

Thus, we have

$$\left(h^2 \int_\Omega \left| \partial_t u(T,x) e^{-\frac{\varphi(T,x)}{h}} \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \leq C\sqrt{h}.$$

Therefore

$$h \int_\Omega \partial_t u(T,x)\overline{v(T,x)}\mathrm{d}x \to 0 \text{ as } h \to 0^+.$$

For $\varepsilon > 0$, define

$$\partial\Omega_{+,\varepsilon,\omega} = \{x \in \partial\Omega : \nu(x) \cdot \omega > \varepsilon\}.$$

and

$$\Sigma_{+,\varepsilon,\omega} = (0,T) \times \partial\Omega_{+,\varepsilon,\omega}.$$

Next we prove (31). Since $\Sigma \backslash G \subseteq \Sigma_{+,\varepsilon,\omega}$ for all ω such that $|\omega - \omega_0| \leq \varepsilon$, substituting $v = v_d$ from (19), in (31) we have

$$\begin{aligned}
&\left| \int\limits_{\Sigma\backslash G} \partial_\nu u(t,x) v(t,x) \mathrm{d}S_x \mathrm{d}t \right| \leq \int\limits_{\Sigma+,\varepsilon,\omega} \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \left(B_d + hR_d\right)(t,x) \right| \mathrm{d}S_x \mathrm{d}t \\
&\leq C \left(1 + \|hR_d\|^2_{L^2(\Sigma)}\right)^{\frac{1}{2}} \left(\int\limits_{\Sigma+,\varepsilon,\omega} \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t \right)
\end{aligned}$$

with $C > 0$ is independent of h and this inequality holds for all ω such that $|\omega - \omega_0| \leq \varepsilon$. Next by trace theorem, we have that $\|R_d\|_{L^2(\Sigma)} \leq C \|R_d\|_{H^1_{\mathrm{scl}(Q)}}$.

Using this, we get

$$\left| \int\limits_{\Sigma\backslash G} \partial_\nu u(t,x) v(t,x) \mathrm{d}S_x \mathrm{d}t \right| \leq C \left(\int\limits_{\Sigma+,\varepsilon,\omega} \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t \right)^{\frac{1}{2}}.$$

Now

$$\begin{aligned}
\int\limits_{\Sigma_+,\varepsilon,\omega} \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t &= \frac{1}{\varepsilon} \int\limits_{\Sigma_+,\varepsilon,\omega} \varepsilon \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t \\
&\leq \frac{1}{\varepsilon} \int\limits_{\Sigma_+,\varepsilon,\omega} \partial_\nu \varphi \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t.
\end{aligned}$$

Using the boundary Carleman estimate (3.1), we have

$$\frac{h}{\varepsilon} \int\limits_{\Sigma_+,\varepsilon,\omega} \partial_\nu \varphi \left| \partial_\nu u(t,x) e^{-\frac{\varphi}{h}} \right|^2 \mathrm{d}S_x \mathrm{d}t \leq C \| h e^{-\varphi/h} \mathcal{L}_{\mathcal{A}^{(1)},q_1} u \|^2_{L^2(Q)}.$$

We now proceed as before to conclude that

$$h \int\limits_{\Sigma\backslash G} \partial_\nu u(t,x) \overline{v(t,x)} \mathrm{d}S_x \mathrm{d}t \to 0 \text{ as } h \to 0^+.$$

□

6. Proof of Theorem 2.3

In this section, we prove the uniqueness results.

6.1. Recovery of vector potential $\mathcal{A}$

We consider the integral,

$$\int_Q \left(-2A\cdot\nabla_x u_2 + 2A_0\partial_x u_2 + \widetilde{q}u_2\right)(t,x)\overline{v(t,x)}\mathrm{d}x\mathrm{d}t.$$

Substituting (22) for u_2 and (19) for v into the above equation, and letting $h\to 0^+$, we arrive at

$$\int_Q \left(-\omega\cdot A + A_0\right)\overline{B_d(t,x)}B_g(t,x)\mathrm{d}x\mathrm{d}t = 0 \text{ for all } \omega\in\mathbb{S}^{n-1} \text{ such that } |\omega-\omega_0|\le\varepsilon.$$

Denote $\widetilde{\omega} := (1,-\omega)$, $\mathcal{A} = (A_0, A)$, and using the expressions for B_d and B_g, see (19) and (22), we get

$$J := \int_{\mathbb{R}^{1+n}} \widetilde{\omega}\cdot\mathcal{A}(t,x)e^{-i\xi\cdot(t,x)}\exp\left(\int_0^\infty \widetilde{\omega}\cdot\mathcal{A}(t+s,x-s\omega)\right)\mathrm{d}x\mathrm{d}t = 0,$$

where $\xi\cdot(1,-\omega)=0$ for all ω with $|\omega-\omega_0|<\varepsilon$. We decompose $\mathbb{R}^{1+n} = \mathbb{R}(1,-\omega)\oplus(1,-\omega)^\perp$. We then get

$$J = \int_{(1,-\omega)^\perp} e^{-\mathrm{i}\xi\cdot k}\left(\int_{\mathbb{R}}\widetilde{\omega}\cdot\mathcal{A}(k+\tau(1,-\omega))\exp\left(\int_\tau^\infty \widetilde{\omega}\cdot\mathcal{A}(k+s(1,-\omega))\mathrm{d}s\right)\sqrt{2}\mathrm{d}\tau\right)\mathrm{d}k.$$

Here $\mathrm{d}k$ is the Lebesgue measure on the hyperplane $(1,-\omega)^\perp$.

$$\begin{aligned}
&= -\sqrt{2}\int_{(1,-\omega)^\perp} e^{-\mathrm{i}\xi\cdot k}\left(\int_{\mathbb{R}}\partial_\tau\exp\left(\int_\tau^\infty \widetilde{\omega}\cdot\mathcal{A}(k+s(1,-\omega))\mathrm{d}s\right)\mathrm{d}\tau\right)\mathrm{d}k\\
&= -\sqrt{2}\int_{(1,-\omega)^\perp} e^{-i\xi\cdot k}\left(1-\exp\left(\int_{\mathbb{R}}\widetilde{\omega}\cdot\mathcal{A}(k+s(1,-\omega))\mathrm{d}s\right)\right)\mathrm{d}k\\
&= C\mathcal{F}_{(1,-\omega)^\perp}\left(1-\exp\left(\int_{\mathbb{R}}\widetilde{\omega}\cdot\mathcal{A}(k+s(1,-\omega))\mathrm{d}s\right)\right)(\xi) \text{ for some constant } C.
\end{aligned}$$

Since the integral $J = 0$, we have that

$$\mathcal{F}_{(1,-\omega)^\perp}\left(1 - \exp\left(\int_{\mathbb{R}} \widetilde{\omega}\cdot\mathcal{A}(k + s(1,-\omega))\mathrm{d}s\right)\right)(\xi) = 0,\ k \in (1,-\omega)^\perp.$$

This gives us

$$\exp\left(\int_{\mathbb{R}} \widetilde{\omega}\cdot\mathcal{A}(k + s(1,-\omega))\mathrm{d}s\right) = 1,\ \text{for all}\ \ k \in (1,-\omega)^\perp \text{ and all } \omega \text{ with } |\omega - \omega_0| < \varepsilon.$$

Thus we deduce that

$$\int_{\mathbb{R}} \widetilde{\omega}\cdot\mathcal{A}(t + s, x - s\omega)\mathrm{d}s = 0, \quad (t,x) \in (1,-\omega)^\perp \text{ and for all } \omega \text{ with } |\omega - \omega_0| < \varepsilon. \tag{32}$$

Now we show that the orthogonality condition $(t,x) \in (1,-\omega)^\perp$, can be removed using a change of variables as used in [25].

Consider any $(t,x) \in \mathbb{R}^{1+n}$. Then

$$\left(\frac{t + x\cdot\omega}{2}, x + \frac{(t - x\cdot\omega)\omega}{2}\right)$$

is a point on $(1,-\omega)^\perp$, and we have

$$\int_{\mathbb{R}} \widetilde{\omega}\cdot\mathcal{A}\left(\frac{t + x\cdot\omega}{2} + \widetilde{s}, x + \frac{(t - x\cdot\omega)\omega}{2} - \widetilde{s}\omega\right)\mathrm{d}\widetilde{s} = 0.$$

Consider the following change of variable in the above integral:

$$s = \frac{x\cdot\omega - t}{2} + \widetilde{s}.$$

Then we have

$$\int_{\mathbb{R}} \widetilde{\omega}\cdot\mathcal{A}\left(t + s, x - s\omega\right)\mathrm{d}s = 0.$$

Therefore, we have that

$$\int_{\mathbb{R}} (1,-\omega)\cdot\mathcal{A}(t + s, x - s\omega)\mathrm{d}s = 0 \text{ for all } (t,x) \in \mathbb{R}^{1+n} \text{ and for all } \omega \text{ with } |\omega - \omega_0| < \varepsilon.$$

To conclude the uniqueness result for $\mathcal{A}$, we prove the following lemma.

Lemma 6.1. *Let $n \geq 3$ and $F = (F_0, F_1, \cdots, F_n)$ be a real-valued vector field whose components are $C_c^\infty(R^{1+n})$ functions. Suppose*

$$LF(t,x,\omega) := \int_{\mathbb{R}} (1,\omega) \cdot F(t+s, x+s\omega) \mathrm{d}s = 0$$

for all $\omega \in \mathbb{S}^{n-1}$ near a fixed $\omega_0 \in \mathbb{S}^{n-1}$ and for all $(t,x) \in \mathbb{R}^{1+n}$. Then there exist a $\Phi \in C_c^\infty(\mathbb{R}^{1+n})$ such that $F(t,x) = \nabla_{t,x}\Phi(t,x)$.

Proof. The proof follows the analysis similar to the ones used in [21,29], where support theorems involving light ray transforms have been proved.

Denote $\omega = (\omega^1, \cdots, \omega^n) \in \mathbb{S}^{n-1}$. We write

$$LF(t,x,\omega) = \int_{\mathbb{R}} \sum_{i=0}^{n} \omega^i F_i(t+s, x+s\omega) \mathrm{d}s, \text{ where } \omega^0 = 1.$$

Let $\eta = (\eta_0, \eta_1, \cdots, \eta_n) \in \mathbb{R}^{n+1}$ be arbitrary. We have

$$(\eta \cdot \nabla_{t,x}) LF(t,x,\omega) = \int_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j \partial_j F_i(t+s, x+s\omega) \mathrm{d}s. \tag{33}$$

By fundamental theorem of calculus, we have

$$\int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}s} (\eta \cdot F)(t+s, x+s\omega) \mathrm{d}s = 0.$$

But

$$\frac{\mathrm{d}}{\mathrm{d}s} (\eta \cdot F)(t+s, x+s\omega) = \sum_{i,j=0}^{n} \omega^i \eta_j \partial_i F_j(t+s, x+s\omega).$$

with $\partial_0 = \partial_t$ and $\partial_j = \partial_{x_j}$ for $j = 1, 2, \cdots, n$. Therefore

$$\int_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j \partial_i F_j(t+s, x+s\omega) \mathrm{d}s = 0. \tag{34}$$

Subtracting (34) from (33), we get,

$$(\eta \cdot \nabla_{t,x})\, LF(t,x,\omega) = \int_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j \left(\partial_j F_i - \partial_i F_j\right)(t+s, x+s\omega) \mathrm{d}s.$$

Since $LF(t,x,\omega) = 0$ for all ω near ω_0, and for all $(t,x) \in \mathbb{R}^{1+n}$, we have,

$$Ih(t,x,\omega,\eta) := \int_{\mathbb{R}} \sum_{i,j=0}^{n} \omega^i \eta_j h_{ij}(t+s, x+s\omega) \mathrm{d}s = 0. \tag{35}$$

Next we will show that the $(n+1)$ dimensional Fourier transform $\widehat{h}_{ij}(\zeta) = 0$ for all space-like vectors ζ near the set $\{\zeta : \ \zeta \cdot (1,\omega) = 0, \omega \text{ near } \omega_0\}$.

We have

$$\omega^i \eta_j \widehat{h}_{ij}(\zeta) = \int\limits_{\mathbb{R}^{1+n}} e^{-\mathrm{i}(t,x)\cdot\zeta} \omega^i \eta_j h_{ij}(t,x) \mathrm{d}t\mathrm{d}x,$$

where ω, η are fixed and $\zeta \in (1,\omega)^\perp$. Decomposing

$$\mathbb{R}^{1+n} = \mathbb{R}(1,\omega) + k, \text{ where } k \in (1,\omega)^\perp,$$

we get,

$$\omega^i \eta_j \widehat{h}_{ij}(\zeta) = \sqrt{2} \int\limits_{(1,\omega)^\perp} e^{-\mathrm{i}k\cdot\zeta} \int\limits_{\mathbb{R}} \omega^i \eta_j h_{ij}(k + s(1,\omega)) \, \mathrm{d}s\mathrm{d}k.$$

Using (35), we get that,

$$\sum_{i,j=0}^{n} \omega^i \eta_j \widehat{h}_{ij}(\zeta) = 0, \text{ for all } \zeta \in (1,\omega)^\perp, \text{ for all } \eta \in \mathbb{R}^{1+n}, \text{ and for all } \omega \text{ near } \omega_0 \text{ with } \omega^0 = 1. \tag{36}$$

Let us take for η the standard basis vectors in $\mathbb{R}^{1+n}$.

Now $\{e_j, 1 \le j \le n\}$ be the standard basis of $\mathbb{R}^n$. Let $\omega_0 = e_1$, and assume that $\zeta^0 = (0, e_2)$. Then this is a space-like vector that satisfies the condition $\zeta^0 \in (1,\omega_0)^\perp$. We will show that $\widehat{h}_{ij}(\zeta^0) = 0$ for all $0 \le i,j \le n$. Consider the collection of the following unit vectors for $3 \le k \le n$:

$$\omega_k(\alpha) = \cos(\alpha) e_1 + \sin(\alpha) e_k.$$

Note that for each $3 \le k \le n, \omega_k(\alpha)$ is near ω_0 for α near 0. Also $\zeta^0 \in (1,\omega_k(\alpha))^\perp$ for all such α and for all $3 \le k \le n$. Let η be the collection of standard basis vectors in $\mathbb{R}^{1+n}$.

Substituting the above vectors $\omega_k(\alpha)$ and η, we get the following equations:

$$\widehat{h}_{0j}(\zeta^0) + \cos(\alpha)\widehat{h}_{1j}(\zeta^0) + \sin(\alpha)\widehat{h}_{kj}(\zeta^0) = 0 \text{ for all } 3 \le k \le n, 0 \le j \le n \text{ and } \alpha \text{ near } 0.$$

From this we have that

$$\widehat{h}_{0j}(\zeta^0) = \widehat{h}_{1j}(\zeta^0) = \widehat{h}_{kj}(\zeta^0) = 0 \text{ for all } 3 \le k \le n \text{ and } 0 \le j \le n.$$

Using the fact that

$$\widehat{h}_{ij}(\zeta^0) = -\widehat{h}_{ji}(\zeta^0) \text{ for all } 0 \le i,j \le n,$$

we get

$$\widehat{h}_{ij}(\zeta^0) = 0, \text{ for all } 0 \le i,j \le n \text{ with } \zeta^0 = (0,e_2). \tag{37}$$

Now our goal is to show that the Fourier transform $\widehat{h}_{ij}(\zeta)$ vanishes for all light-like vectors ζ in a small enough neighborhood of $(0, e_2)$. To show this, assume $\zeta = (\tau, \xi)$ be a space-like vector close to $(0, e_2)$. Without loss of generality, assume that ζ is of the form

$$\zeta = (\tau, \xi) \text{ where } |\xi| = 1 \text{ and } |\tau| < 1.$$

For instance, we can take ζ as

$$\zeta = (-\sin\varphi, \xi),$$

where φ is close to 0 and $\xi = (\xi_1, \cdots, \xi_n)$ is written in spherical coordinates as

$$\begin{aligned}
\xi_1 &= \sin\varphi_1 \cos\varphi_2 \\
\xi_2 &= \cos\varphi_1 \\
\xi_3 &= \sin\varphi_1 \sin\varphi_2 \cos\varphi_3 \\
&\vdots \\
\xi_{n-1} &= \sin\varphi_1 \cdots \sin\varphi_{n-2} \cos\varphi_{n-1} \\
\xi_{n-1} &= \sin\varphi_1 \cdots \sin\varphi_{n-2} \sin\varphi_{n-1}.
\end{aligned}$$

Note that if $\varphi_1, \cdots, \varphi_{n-1}$ are close to 0, then ξ is close to e_2.

Let

$$\omega_\varphi = \cos(\varphi)e_1 + \sin(\varphi)e_2.$$

Also let

$$\omega_\varphi^k(\alpha) = \cos(\alpha)\cos(\varphi)e_1 + \sin(\varphi)e_2 + \sin(\alpha)\cos(\varphi)e_k \text{ for } k \geq 3.$$

Since φ is close to 0, and letting α close enough to 0, we have that ω_φ and ω_φ^k are close to ω_0.

Now, let A be an orthogonal transformation such that $Ae_2 = \xi$, where ξ is as above. With this A, consider the vectors

$$\omega_\zeta = A\omega_\varphi \text{ and } \omega_\zeta^k = A\omega_\varphi^k \text{ for } k \geq 3.$$

We have that

$$\zeta \in (1, \omega_\zeta)^\perp.$$

To see this, note that $\zeta = (-\sin\varphi, \xi)$ and we have

$$-\sin\varphi + \langle Ae_2, A\omega_\varphi \rangle = -\sin\varphi + \langle e_2, \omega_\varphi \rangle = 0.$$

Also similarly, we have that,

$$\zeta \in (1, \omega_\zeta^k)^\perp \text{ for all } k \geq 3.$$

Using the standard basis vectors for η_j and the vectors $(1, \omega_\zeta^k)$ in (36), we get, for all $k \geq 3$,

$$\widehat{h}_{0j}(\zeta)+\sin\varphi\left(\sum_{i=1}^{n} a_{i2}\widehat{h}_{ij}(\zeta)\right)+\cos\alpha\cos\varphi\left(\sum_{i=1}^{n} a_{i1}\widehat{h}_{ij}(\zeta)\right)+\sin\alpha\cos\varphi\left(\sum_{i=1}^{n} a_{ik}\widehat{h}_{ij}(\zeta)\right)=0$$

This then implies that

$$\widehat{h}_{0j}(\zeta)+\sin\varphi\left(\sum_{i=1}^{n} a_{i2}\widehat{h}_{ij}(\zeta)\right)=0$$
$$\cos\varphi\left(\sum_{i=1}^{n} a_{i1}\widehat{h}_{ij}(\zeta)\right)=0$$
$$\cos\varphi\left(\sum_{i=1}^{n} a_{ik}\widehat{h}_{ij}(\zeta)\right)=0$$

Letting $j = 0$, since $\widehat{h}_{00}(\zeta) = 0$, we have,

$$\sum_{i=1}^{n} a_{ij}\widehat{h}_{i0}(\zeta) = 0 \text{ for all } 1 \leq j \leq n.$$

Since A is an invertible matrix, we have that $\widehat{h}_{i0}(\zeta) = 0$ for all $1 \leq i \leq n$. Now proceeding similarly and using the fact that h_{ij} is alternating, we have that $\widehat{h}_{ij}(\zeta) = 0$ for all $0 \leq i, j \leq n$. The same argument as above also works for $r\zeta$, where ζ is as above and $r > 0$.

Since the support of all h_{ij} is a compact subset of $\mathbb{R}^{1+n}$, by Paley-Wiener theorem, we have $\widehat{h}_{ij}(\zeta) = 0 \ \forall \ i, j = 0, 1, 2, ..., n$. Hence, by Fourier inversion formula, we see that $h_{ij}(t, x) = 0 \ \forall \ (t, x) \in \mathbb{R}^{1+n}$, this gives us $\nabla_{t,x} F(t, x) = 0 \ \forall \ (t, x) \in \mathbb{R}^{1+n}$. Using Poincaré lemma, we have that there exists a $\Phi(t, x) \in C_c^\infty(\mathbb{R}^{1+n})$ such that $F = \nabla_{t,x}\Phi$.

This then gives that

$$\mathcal{A}(t, x) := \left(\mathcal{A}^{(2)} - \mathcal{A}^{(1)}\right)(t, x) := (A_0, A_1, \cdots, A_n) := \nabla_{t,x}\Phi(t, x).$$

□

6.2. Recovery of potential q

In Section 6.1, we showed that there exist a Φ such that $(\mathcal{A}_2 - \mathcal{A}_1)(t, x) = \nabla_{t,x}\Phi(t, x)$. After replacing the pair $(\mathcal{A}^{(1)}, q_1)$ by $(\mathcal{A}^{(3)}, q_3)$ where $\mathcal{A}^{(3)} = \mathcal{A}^{(1)} + \nabla_{t,x}\Phi$ and $q_3 = q_1$, we conclude that $\mathcal{A}^{(3)} = \mathcal{A}^{(2)}$. Therefore substituting (22) for u_2 and (19) for v, and letting $h \to 0^+$ in

$$\int_Q q(t, x)u_2(t, x)v(t, x)\mathrm{d}x\mathrm{d}t = \int_\Omega \partial_t u(T, x)v(T, x)\mathrm{d}x - \int_{\Sigma\setminus G} \partial_\nu u(t, x)\overline{v(t, x)}\mathrm{d}S_x\mathrm{d}t,$$

we get,

$$\int_{\mathbb{R}^{1+n}} q(t,x)e^{-\mathrm{i}\xi\cdot(t,x)}\mathrm{d}x\mathrm{d}t = 0 \text{ for all } \xi \in (1,-\omega)^{\perp} \text{ and } \omega \text{ near } \omega_0.$$

The set of all ξ such that $\xi \in (1,-\omega)^{\perp}$ for ω near ω_0 forms an open cone and since $q \in L^{\infty}(Q)$ has compact support, using Paley-Wiener theorem we conclude that $q_1(t,x) = q_2(t,x)$ for all $(t,x) \in Q$. This completes the proof of Theorem 2.3.

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