

A Support Theorem for the Geodesic Ray Transform on Functions

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Abstract Let (M, g) be a simple Riemannian manifold. Under the assumption that the metric g is real-analytic, it is shown that if the geodesic ray transform of a function $f \in L^2(M)$ vanishes on an appropriate open set of geodesics, then $f = 0$ on the set of points lying on these geodesics. The approach is based on analytic microlocal analysis.

Keywords Geodesic ray transform · Support theorems · Microlocal analysis

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1 Introduction

Let $(M, \partial M, g)$ be a smooth n -dimensional compact Riemannian manifold with boundary. In this paper we study the geodesic ray transform on functions. This is defined as follows: Let $\gamma : [0, l(\gamma)] \rightarrow M$ be a geodesic joining boundary points, where $l(\gamma)$ is the length of this geodesic. The geodesic ray transform I of a function f along γ is defined as

$$If(\gamma) = \int_0^{l(\gamma)} f(\gamma(t))dt.$$

We address the question whether the geodesic ray transform of a function $f \in L^2(M)$ over an appropriate open set \mathcal{A} of geodesics uniquely determines the function f on the set of points lying on the geodesics of \mathcal{A} . The results of this kind are known as support theorems in integral geometry literature.

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One needs additional restrictions on the metric even to prove injectivity results for this transform as the following counterexample shows [16]: Consider the unit sphere with a small disk excised out from its east pole making the resulting manifold a smooth manifold with boundary. Now consider a function $f = 1$ on a small disk centered at the north pole and $f = -1$ on a symmetrical disk centered at the south pole. Then the geodesic ray transform vanishes identically, but f is not 0.

One assumption is to assume that the Riemannian manifold $(M, \partial M, g)$ is simple. A compact Riemannian manifold with boundary $(M, \partial M, g)$ is simple if

- (a) The boundary ∂M is strictly convex: $\langle \nabla_{\xi} \nu, \xi \rangle > 0$ for $\xi \in T_x(\partial M)$. Here ν is the unit outward normal to the boundary.
- (b) The map $\exp_x : \exp_x^{-1} M \rightarrow M$ is a diffeomorphism for each $x \in M$.

It is well known that simple Riemannian manifolds are diffeomorphic to closed balls in \mathbb{R}^n [13]. So from now on, without loss of generality, we will assume that our manifold M is a closed ball in \mathbb{R}^n .

It is known from the work of Mukhometov that on a simple Riemannian manifold, the geodesic ray transform is injective. We refer to Sharafutdinov's book [13] for this and other results related to the geodesic ray transform. However a support theorem for this transform has been an open question. In this paper, under the additional assumption of real-analyticity of the simple Riemannian metric, we prove a support theorem. We use analytic microlocal analysis to prove our results. The tools of microlocal analysis were introduced in the Radon transform setting by V. Guillemin. See [9] for more details. Based on this work, Boman and Quinto in [3] proved a support theorem for the generalized Radon transform integrated against real analytic weights by using a microlocal version of unique continuation of analytic functions due to Kawai-Kashiwara-Hörmander. Our support theorem relies on this unique continuation result.

There are several support theorems in integral geometry based on the application of the aforementioned result of Kawai-Kashiwara-Hörmander. We refer the reader to the following papers for some of the references [1–4, 6, 8, 11, 12, 18]. The geometric considerations and the transform that we study in this paper makes our result different from these earlier known support theorems.

We make a remark before stating our main result. If (M, g) is a simple Riemannian manifold with g real-analytic, we can extend the manifold slightly to a larger manifold \tilde{M} with M contained in the interior of \tilde{M} , and the metric g real-analytically to \tilde{M} (we call the extended metric \tilde{g}) such that (\tilde{M}, \tilde{g}) is simple. From now on, we will assume that \tilde{M} is a simple manifold as above. Also when we talk about geodesics, we will assume that they are maximal geodesics joining boundary points.

We now state our main result.

Theorem 1 *Let $(M, \partial M, g)$ be a simple Riemannian manifold and assume that the metric g is real-analytic. Let \mathcal{A} be an open set of geodesics in \tilde{M} such that each geodesic $\gamma \in \mathcal{A}$ can be deformed to a point on the boundary $\partial \tilde{M}$ by geodesics in \mathcal{A} . Let $M_{\mathcal{A}}$ be the set of points lying on the intersection of these geodesics with M . If $f \in L^2(M)$ is a function such that $I f(\gamma) = 0$ for each geodesic γ in M such that its geodesic extension to \tilde{M} belongs to \mathcal{A} , then $f = 0$ on $M_{\mathcal{A}}$.*

Note that the function is a priori defined only on M . We will extend f to \tilde{M} such that $f = 0$ in $\tilde{M} \setminus M$. Then we will still have $If(\gamma) = 0$ for each γ in \mathcal{A} .

2 Analytic Regularity of f along Conormal Directions of $\gamma \in \mathcal{A}$

We fix a geodesic $\gamma_0 \in \mathcal{A}$. Since \mathcal{A} is open, for each γ near γ_0 , we have $If(\gamma) = 0$. Under this assumption, we show that f is microlocally analytic along conormal directions of γ_0 . Our references for analytic microlocal analysis are [10, Chap. 8], [17, Chap. 5] and [14].

Proposition 1 *Let $\gamma_0 \in \mathcal{A}$. Then $WF_a(f) \cap N^*\gamma_0 = \emptyset$.*

Proof The proof of this proposition is fairly straightforward for C^∞ wavefront sets. For, one can consider a C_c^∞ cut-off ψ in a small neighborhood of γ_0 and consider the pseudodifferential operator $I^*\psi I$. This is elliptic in $N^*\gamma_0 \setminus \{0\}$. If we assume that $If = 0$ near γ_0 , then by microlocal elliptic regularity, we get that f is smooth in $N^*\gamma_0 \setminus \{0\}$. The proof for the analytic case is much more involved because with the cut-off ψ , $I^*\psi I$ is not an analytic pseudodifferential operator anymore and there are restrictions on the kind of cut-offs we can use.

In the analytic case, we use a recent result of Stefanov-Uhlmann given in [16, Proposition 2] based on the characterization of analytic singularities using the generalized FBI transform and the complex stationary phase method. See the book of J. Sjöstrand [14] for more details. Actually Stefanov-Uhlmann in [16, Proposition 2] have proved the statement of the proposition with f replaced by a solenoidal symmetric tensor field. Their proof carries over here with straightforward modifications and in fact the arguments for functions are simpler. We will not repeat the proof here, but just mention that the simplification in our case is due to the fact that only one phase function satisfying the estimate in [16, equation 51] is enough to prove our result. This estimate also shows that the conormals to γ_0 are not in the analytic wavefront set of f ; see [14, Definition 6.1]. We also mention a recent work of Frigiyik-Stefanov-Uhlmann in [5] where similar ideas based on the complex stationary phase method are used. □

It is of interest, whether one can prove the statement of Proposition 1 relying on analytic pseudodifferential operator methods alone. We are able to provide a proof for 2-dimensional manifolds. We parameterize the geodesics of M by $\partial_+ SM := \{(x, \xi) : x \in \partial M, |\xi| = 1, \langle \nu, \xi \rangle \leq 0\}$ where ν is the outer unit normal to the boundary. Let γ_0 intersect the boundary ∂M at a point x_0 in the direction ξ_0 such that $(x_0, \xi_0) \in \partial_+ SM$. We now perturb the point x_0 along the boundary and the vector ξ_0 slightly such that the geodesics starting from these perturbed points and directions still lie in \mathcal{A} . Let ψ be a C_c^∞ function supported in this perturbation and identically 1 in a smaller neighborhood of (x_0, ξ_0) . As in [3], we write

$$I^*If = I^*\psi If + I^*(1 - \psi)If.$$

By hypothesis, the first term on the right above is 0. For the second term above, we analyze what I^* does to the analytic singularities of $(1 - \psi)If$. For this we make use of a calculation proved in [7, p. 212, equation 2.15]. Let $Z = \{(x, y, \eta) \in M \times \mathcal{A} : \exp_y(s\eta) = x, (y, \eta) \in \partial SM, s \in \mathbb{R}\}$. (Here SM is the unit sphere bundle.) We then have that N^*Z is given by

$$\begin{aligned} N^*Z = \{ & (\exp_y(s\eta), y, \eta; \\ & ((D_\eta \exp)^*)^{-1}(\eta \lrcorner \Omega), ((D_y \exp)^*(D_\eta \exp)^*)^{-1}(\eta \lrcorner \Omega), s\eta \lrcorner \Omega|_{T\mathcal{A}}) : \\ & (y, \eta) \in \mathcal{A}, s \in \mathbb{R}, \Omega \in \Lambda^2 T_y^*M\}. \end{aligned} \quad (2.1)$$

Now we show that the projection $N^*Z \rightarrow T^*\mathcal{A}$ is injective. Suppose we are given:

$$(y, \eta, (D_y \exp)^*((D_\eta \exp)^*)^{-1}(\eta \lrcorner \Omega), s\eta \lrcorner \Omega|_{T\mathcal{A}}).$$

Then $s((D_y \exp)^*(D_\eta \exp)^*)^{-1}(\eta \lrcorner \Omega) = ((D_y \exp)^*(D_\eta \exp)^*)^{-1}(s\eta \lrcorner \Omega)$. The right hand side is known and the vector part of the left hand side is known. So we can recover s , from which we can recover $\exp_y(s\eta)$.

For simplicity let us denote $u = (1 - \psi)If$. Based on the injectivity of the projection shown above, we have that an analytic singularity at γ_0 pulls back to an analytic singularity at a point on the geodesic γ_0 and no other geodesic. From (2.1), we get that N^*Z does not contain any elements where one of the conormal directions is 0. Now using [10, Theorem 8.5.5], we get that if the projection pulls back to an analytic singularity on the geodesic γ_0 , then such a singular direction has to be conormal to the geodesic γ_0 . Now u is analytic at γ_0 in all directions and in 2 dimensions the singularities of the cut-off ψ pull back to singularities not conormal to the geodesic γ_0 . Therefore we get that $WF_a(I^*If) \cap N^*\gamma_0 = \emptyset$. Now by the work of Stefanov-Uhlmann in [15], we have that I^*I is an elliptic analytic pseudodifferential operator. Therefore using elliptic regularity, we get $WF_a f \cap N^*\gamma_0 = \emptyset$. This provides another proof of the proposition for the 2-dimensional case.

In higher dimensions, this proof does not work because the analytic singularities of $(1 - \psi)$ could appear conormal to a different geodesic γ_1 that intersects γ_0 and it is possible that the analytic singular direction is at the point of intersection and also conormal to both γ_0 and γ_1 . It is an interesting question whether by a very careful choice of the cut-off ψ and some modification of the arguments above, a similar proof can be given for the higher dimensional case.

3 Proof of the Main Result

In this section, we prove the main result. Recall that we have extended the function f to be 0 in $\tilde{M} \setminus M$.

Proof of Theorem 1 Assume first that $n \geq 3$. Fix a geodesic $\gamma \in \mathcal{A}$. We define a continuous map $Q : [0, 1] \rightarrow \mathcal{A}$ such that $Q(1) = \sigma$, where $\sigma \in \mathcal{A}$ is a geodesic that does not intersect $\text{supp } f$ and $Q(0) = \gamma$. Note that by the openness of \mathcal{A} , there is always a geodesic in \mathcal{A} that does not intersect $\text{supp}(f)$. Now consider a geodesic $Q(t)$.

Assume that this geodesic is parameterized by $(x, \xi) \in \partial S_x \tilde{M}$ where x is a point on the boundary ∂M . Now consider all unit vectors that are at a fixed angle to ξ . If we assume this angle is small enough, then the geodesics starting at x in this cone of directions lie in \mathcal{A} and also that there is at least one cone that does not intersect $\text{supp}(f)$. Let us call this cone of geodesics $\mathcal{C}_{Q(t)}$. We do this construction for all $0 \leq t \leq 1$. We define a new map P that associates for every $0 \leq t \leq 1$, the cone $\mathcal{C}_{Q(t)}$. Following [3], we let,

$$t_1 = \inf\{t : P(t_2) \cap \text{supp } f = \emptyset \text{ for all } t_2 > t\}.$$

Assume $t_1 > 0$. Then $P(t_1)$ intersects support of f at the boundary on a compact set. Since the cone $P(t_1)$ is an embedded hypersurface, by [10, Definition 8.5.7], we have that at a point of intersection, the normal to the cone is also normal to $\text{supp } f$. Now by [10, Theorem 8.5.6], such a normal is in $WF_a(f)$. But this is a contradiction to Proposition 1. Therefore $f = 0$ in a neighborhood of this point of intersection. Since the points of intersection is a compact set, by considering a finite number of such points, we can show that $f = 0$ at a positive distance away from this cone. But this contradicts the infimum value t_1 above. Hence $t_1 = 0$ and so we have $f = 0$ on $Q([0, 1])$. For the case of $n = 2$, the geodesics themselves form hypersurfaces and the same argument as above holds without the need for the cone construction. This shows that $f = 0$ on the set of points lying on the deformation. The proof of the theorem is complete. \square

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