MICROLOCAL INVERSION OF A 3-DIMENSIONAL RESTRICTED TRANSVERSE RAY TRANSFORM OF SYMMETRIC m-TENSOR FIELDS

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Abstract. We study the problem of inverting a restricted transverse ray transform to recover a symmetric m-tensor field in $\mathbb{R}^3$ using microlocal analysis techniques. More precisely, we prove that a symmetric m-tensor field can be recovered up to a known singular term and a smoothing term if its transverse ray transform is known along all lines intersecting a fixed smooth curve satisfying the Kirillov-Tuy condition.

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1. Introduction

The study of transverse ray transforms (TRT) of symmetric tensor fields is of interest in problems arising in polarization and diffraction tomography. In this paper, we consider an approximate inversion of a TRT acting on symmetric tensor fields restricted to all lines passing through a fixed curve in $\mathbb{R}^3$. More precisely, we use techniques from microlocal analysis to construct a relative left parametrix for the restricted TRT.

We denote the space of covariant symmetric m-tensors in $\mathbb{R}^3$ by $S^m = S^m(\mathbb{R}^3)$. Let $C_c^\infty(S^m)$ be the space of smooth compactly supported symmetric m-tensor fields in $\mathbb{R}^3$. In $\mathbb{R}^3$, an element $f \in C_c^\infty(S^m)$ can be written as

$$f(x) = f_{i_1\cdots i_m}(x)\mathrm{d}x^{i_1}\cdots \mathrm{d}x^{i_m},$$

with $\{f_{i_1\cdots i_m}(x)\}$ symmetric in its components, smooth and compactly supported. With repeating indices, Einstein summation convention will be assumed throughout this paper.

We now define TRT, the primary object of study in this paper. Let $\omega \in \mathbb{S}^2$ be represented in spherical coordinates by

$$\omega = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2).$$

where $0 \le \theta_1 < \pi$ and $0 \le \theta_2 < 2\pi$.

Given ω, let ω_1 and ω_2 be defined as follows:

$$\begin{aligned}\omega_1 &= (-\sin\theta_1, \cos\theta_1\cos\theta_2, \cos\theta_1\sin\theta_2) \text{ and}\\ \omega_2 &= (0, -\sin\theta_2, \cos\theta_2).\end{aligned} \tag{1}$$

We define the transverse ray transform $\mathcal{T}$ [26] in $\mathbb{R}^3$ as follows:

Definition 1.1. [Transverse ray transform $\mathcal{T}$, [26]] For $0 \le i \le m$, define $\mathcal{T} = (\mathcal{T}_i) : C_c^\infty(S^m) \to \left(C^\infty(T\mathbb{S}^2)\right)^{m+1}$ by

$$\mathcal{T}_i f(x,\omega) = \int_{\mathbb{R}} f_{j_1 j_2\cdots j_m}(x+t\omega)\omega_1^{j_1}\cdots\omega_1^{j_{m-i}}\omega_2^{j_{m-(i-1)}}\cdots\omega_2^{j_m}\mathrm{d}t, \tag{2}$$

In 2-dimensions, TRT and the standard ray transform [26], also called the longitudinal ray transform (LRT), give equivalent information and it is well-known that the latter transform on symmetric tensor fields has an infinite dimensional kernel. Hence it is not possible to reconstruct the tensor field f from its transverse ray transform in 2-dimensions. Furthermore, the space of lines in $\mathbb{R}^n$ is $2n-2$ dimensional, and in dimensions $n \ge 3$, the problem of recovery of f from $\mathcal{T}f$ is over-determined. Therefore a natural question is to investigate the inversion of $\mathcal{T}$ restricted to an n-dimensional data set. We address this incomplete data problem for the case of dimension $n = 3$ in this paper.

The study of inversion of TRT and the corresponding non-linear problem appearing in polarization tomography has been considered in several prior works [26, 24, 27, 14, 21, 7, 6, 18]. With respect to the study of restricted TRT, we refer to the works of [22, 7]. Recently a support theorem for TRT in the setting of analytic simple Riemannian manifolds was considered by [1].

We study the inversion of an incomplete data TRT from a microlocal analysis point of view. We are interested in the reconstruction of singularities of the symmetric tensor field f given its restricted TRT. The study of generalized Radon transforms in the framework of Fourier integral operators began with the fundamental work of Guillemin [11] and Guillemin-Sternberg [12]. Since then, microlocal analysis has become a very powerful tool in the study of tomography problems; see [10, 9, 4, 5, 28, 15, 20, 25, 29, 30, 31, 30, 16, 2]. Of these works, the paper [9] is a fundamental work where Greenleaf and Uhlmann studied an incomplete data ray transform on functions in the setting of Riemannian manifolds. However, most of these works are done for LRT and to the best of our knowledge, other than the support theorem result [1], we are not aware of any prior work that studies a restricted TRT from the view point of microlocal analysis.

Specifically, we study the microlocal inversion of the Euclidean TRT on symmetric m-tensor fields given the incomplete data set consisting of all lines passing through a fixed curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3$. The transverse ray transform $\mathcal{T}$ defined in (2) restricted to lines passing through the curve $\boldsymbol{\gamma}$ will be denoted by $\mathcal{T}_{\boldsymbol{\gamma}}$ and its formal L^2 adjoint by $\mathcal{T}^*_{\boldsymbol{\gamma}}$. We determine the extent to which the wavefront set of a symmetric m-tensor field can be recovered from the wavefront set of its restricted TRT.

The main motivation for our article comes from the related works done for the longitudinal ray transform [9, 19, 20, 25, 17] and we mainly follow the techniques from these works.

The article is organized as follows. In §2, we state the main result, some fundamental results about distributions associated to two cleanly intersecting Lagrangians introduced by [23, 13, 9] and the microlocal results relevant for the analysis of our transform. §3 is devoted to stating some preliminary results about the restricted TRT. We do not give any proofs in this section as all the details follow in a straightforward manner from the works [19, 20, 17]. We prove the main result in §4 and §5.

2. Statement of the main result

In order to invert the TRT restricted to lines passing through a curve in $\mathbb{R}^3$, we need to place some conditions on the curve $\boldsymbol{\gamma}$. We state them and proceed to the main result.

(1) The curve $\boldsymbol{\gamma} : I \to \mathbb{R}^3$, where I is a bounded interval, is smooth, regular and without self-intersections.
(2) There is a uniform bound on the number of intersection points of almost every plane in $\mathbb{R}^3$ with the curve $\boldsymbol{\gamma}$, see [19].
(3) The curve $\boldsymbol{\gamma}$ satisfies the Kirillov-Tuy condition; see Definition 2.1 below.

Definition 2.1 (Kirillov-Tuy condition, [17])**.** Consider a ball B in $\mathbb{R}^3$. We say that a smooth curve $\boldsymbol{\gamma}$ defined on a bounded interval satisfies the Kirillov-Tuy condition of order $m \geq 1$ if for almost all planes H in $\mathbb{R}^3$ intersecting the ball B, there is at least $(m+1)$ points $\boldsymbol{\gamma}(t_1), \dots, \boldsymbol{\gamma}(t_{m+1})$, in the intersection of the plane and the curve $\boldsymbol{\gamma}$, such that for almost all $x \in H \cap B$ any two vectors in the collection $\{(x - \boldsymbol{\gamma}(t_i)), 1 \leq i \leq m+1\}$ is linearly independent.

Remark 2.2. In dimension $n = 3$, the Kirillov-Tuy condition is equivalent to the collection of vectors consisting of the m^{th} symmetric tensor product $\{(x - \boldsymbol{\gamma}(t_i)^{\odot m}; 1 \leq i \leq m+1\}$ being linearly independent. However, this is not the case for higher dimensions in general [17]. Due to this reason, the above mentioned definition of Kirillov-Tuy condition is not sufficient to microlocally invert the restricted TRT. It is an interesting question to define the appropriate Kirillov-Tuy condition for the inversion of TRT in higher dimensions.

Following [9, 25], we now define the following sets. The definitions of these sets is motivated by the fact that these are the wavefront set directions that we can recover based on microlocal analysis techniques.

Let B be the ball that appears in the definition of Kirillov-Tuy condition. Denote the plane passing through x and perpendicular to ξ by $x + \xi^{\perp}$.

Let

$$\Xi = \Big\{(x,\xi) \in T^*B \setminus \{0\} : \text{ there exists at least } m+1 \text{ directions from } x \text{ to } (x+\xi^{\perp}) \cap \boldsymbol{\gamma} \text{ and any two of them are linearly independent}\Big\}.$$

(3)
$$\Xi_{\Delta} = \Big\{(x,\xi) \in \Xi : x+\xi^{\perp} \text{ intersects } \boldsymbol{\gamma} \text{ transversely}\Big\}.$$

$$\Xi_{\Lambda} = \Big\{(x,\xi) \in \Xi : x+\xi^{\perp} \text{ is tangent (only at finite number of points) at (say) } \{\boldsymbol{\gamma}(t_1), \cdots, \boldsymbol{\gamma}(t_N)\} \text{ and } \langle \boldsymbol{\gamma}''(t_i), \xi\rangle \neq 0 \text{ for } i = 1,2,\cdots,N\Big\}.$$

We now state the main result.

Theorem 2.3. *Let $\Xi_0 \subseteq \Xi_{\Delta}$ be such that $\overline{\Xi}_0 \subseteq \Xi_{\Delta} \cup \Xi_{\Lambda}$ and K be a closed conic subset of Ξ_0. Let $\mathcal{E}'_K(B) \subset \mathcal{E}'(B)$ denote the space of compactly supported distributions in B whose wavefront set is contained in K. Then there exists an operator $\mathcal{B} \in I^{0,1}(\Delta, \Lambda)$ and an operator $\mathcal{A} \in I^{-1/2}(\Lambda)$ such that for any symmetric m-tensor field f with coefficients in $\mathcal{E}'_K(B)$,*

$$\mathcal{B}\mathcal{T}_{\boldsymbol{\gamma}}^*\mathcal{T}_{\boldsymbol{\gamma}} f = f + \mathcal{A}f + \textit{smoothing terms.}$$

For the definition of $I^{p,l}$ classes, we refer the reader to the three seminal works on this subject [23, 13, 9]. For the convenience of the reader, we give a quick summary of the properties of the $I^{p,l}$ class of distributions [13]. Let $u \in I^{p,l}(\Delta,\Lambda)$, where Δ and Λ are two cleanly intersecting Lagrangians with intersection Σ. Then

(1) $WF(u) \subset \Delta \cup \Lambda$.
(2) Microlocally, the Schwartz kernel of u equals the Schwartz kernel of a pseudodifferential operator of order $p+l$ on $\Delta \setminus \Lambda$ and that of a classical Fourier integral operator of order p on $\Lambda \setminus \Delta$.
(3) $I^{p,l} \subset I^{p',l'}$ if $p \leq p'$ and $l \leq l'$.
(4) $\cap_l I^{p,l}(\Delta,\Lambda) \subset I^p(\Lambda)$.
(5) $\cap_p I^{p,l}(\Delta,\Lambda) \subset$ The class of smoothing operators.
(6) The principal symbol $\sigma_0(u)$ on $\Delta \setminus \Sigma$ has the singularity on Σ as a conormal distribution of order $l - \frac{k}{2}$, where k is the codimension of Σ as a submanifold of Δ or Λ.
(7) If the principal symbol $\sigma_0(u) = 0$ on $\Delta \setminus \Sigma$, then $u \in I^{p,l-1}(\Delta,\Lambda) + I^{p-1,l}(\Delta,\Lambda)$.
(8) u is said to be elliptic if the principal symbol $\sigma_0(u) \neq 0$ on $\Delta \setminus \Sigma$ if $k \geq 2$, and for $k = 1$, if $\sigma_0(u) \neq 0$ on each connected component of $\Delta \setminus \Sigma$.

The Lagrangian Λ defined in (8) arises as a flowout, and the main tool in the construction of a relative left parametrix for our operator $\mathcal{T}_{\boldsymbol{\gamma}}^*\mathcal{T}_{\boldsymbol{\gamma}}$ is the following composition calculus due to Antoniano and Uhlmann [3]:

Theorem 2.4 ([3]). *If $A \in I^{p,l}(\Delta,\Lambda)$ and $B \in I^{p',l'}(\Delta,\Lambda)$, then composition of A and B, $A \circ B \in I^{p+p'+\frac{k}{2}, l+l'-\frac{k}{2}}(\Delta,\Lambda)$ and the prinicipal symbol, $\sigma_0(A \circ B) = \sigma_0(A)\sigma_0(B)$, where, k is the codimension of Σ as a submanifold of either Δ or Λ.*

We prove the theorem by adopting the strategy of [9, 20, 25, 17] in the TRT setting. More precisely, we compute the principal symbol of the operator $\mathcal{T}_{\boldsymbol{\gamma}}^*\mathcal{T}_{\boldsymbol{\gamma}}$ on the diagonal Δ away from the set Σ and use this principal symbol to construct a relative left parametrix for this operator. Since we deal with a restricted transverse ray transform, the inversion procedure introduces an additional error term (in addition to smoothing terms), but this error term is a Fourier integral operator associated to the known Lagrangian Λ.

3. Preliminary results

In this section, we state some preliminary results regarding the singularities of the left and right projections from the canonical relation associated to the operator $\mathcal{T}_{\boldsymbol{\gamma}}$. The proofs follow by straightforward modifications of the ones given in [19, 17] and therefore we skip them.

Let us denote by $\mathcal{C}$, the line complex consisting of all lines passing through the curve $\boldsymbol{\gamma}$. Let l denote a line in our line complex $\mathcal{C}$ and

$$Z = \{(l,x) : x \in l\} \subset \mathcal{C} \times \mathbb{R}^n$$

be the point-line relation. We have that (t,ω,s) is a local parametrization of Z. The conormal bundle of Z, N^*Z, is described by $\{(t,\omega,s,\Gamma,\xi)\}$ where

$$\xi = z_1\omega_1 + z_2\omega_2 \text{ for some } z_1 \text{ and } z_2 \in \mathbb{R}, \tag{4}$$

and ω_i's are given by (1), and

$$\Gamma = \begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\end{pmatrix} = \begin{pmatrix}-\xi\cdot\gamma'(t)\\ -sz_1\\ -sz_2\sin\theta_1\end{pmatrix}. \tag{5}$$

Lemma 3.1. *The map*

$$\Phi : (t,\theta_1,\theta_2,s,z_1,z_2) \to (t,\theta_1,\theta_2,\Gamma;x,\xi)$$

with Γ as in (5), ξ as in (4) and $x = \gamma(t)+s\omega$ gives a local parametrization of N^*Z at the points where $\theta_1 \neq 0,\pi$.

Proposition 3.2. *Each component of the operator $\mathcal{T}_\gamma$ is a Fourier integral operator of order $-1/2$ with the associated canonical relation C given by $(N^*Z)'$ where $Z = \{(l,x) : x\in l\}$. The left and the right projections π_L and π_R from C drop rank simply by 1 on the set*

$$\Sigma := \{(t,\theta_1,\theta_2,s,z_1,z_2) : \gamma'(t)\cdot\xi = 0\}, \tag{6}$$

where ξ is given by (4). The left projection π_L has a blowdown singularity along Σ and the right projection π_R has a fold singularity along Σ.

We refer the reader to [8] for the definitions of fold and blowdown singularities.

Lemma 3.3. *The wavefront set of the Schwartz kernel of $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ satisfies the following:*

$$WF(\mathcal{T}_\gamma^*\mathcal{T}_\gamma) \subset \Delta\cup\Lambda,$$

where Δ and Λ are defined as follows:

$$\Delta = \left\{(x,\xi;x,\xi) : x = \gamma(t)+s\theta, \xi\in\theta^\perp\setminus\{0\}\right\} \text{ and} \tag{7}$$

$$\Lambda = \left\{(x,\xi,y,\frac{\tau}{\tilde\tau}\xi) : x = \gamma(t)+\tau\theta, y = \gamma(t)+\tilde\tau\theta, \xi\in\theta^\perp\setminus\{0\}, \gamma'(t)\cdot\xi = 0, \tau\neq 0, \tilde\tau\neq 0\right\}. \tag{8}$$

Furthermore, Δ and Λ intersect cleanly due to the third condition in (3).

Lemma 3.4. [19] *The Lagrangian Λ defined in (8) arises as a flowout from the set $\pi_R(\Sigma)$.*

4. Principal symbol of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$

In this section, we compute the principal symbol matrix of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ and show that it is elliptic. The operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ can be written as

$$\mathcal{T}_\gamma^*\mathcal{T}_\gamma = \sum_{i=0}^{m}\left[R_\gamma^*\left(\omega_1^{j_1}\cdots\omega_1^{j_{m-i}}\omega_2^{j_{m-(i-1)}}\cdots\omega_2^{j_m}\omega_1^{l_1}\cdots\omega_1^{l_{m-i}}\omega_2^{l_{m-(i-1)}}\cdots\omega_2^{l_m}\right)R_\gamma\right],$$

where $\mathcal{R}_\gamma$ is the restricted scalar ray transform and $\mathcal{R}_\gamma^*$ is its formal L^2 adjoint.

Proposition 4.1. *The principal symbol matrix $A_0(x,\xi)$ of the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ is*

$$A_0(x,\xi) = \sum_k\sum_{i=0}^{m}\frac{2\pi\omega_1^{j_1}(t_k)\cdots\omega_1^{j_{m-i}}(t_k)\omega_2^{j_{m-(i-1)}}\cdots\omega_2^{j_m}(t_k)\omega_1^{l_1}(t_k)\cdots\omega_1^{l_{m-i}}(t_k)\omega_2^{l_{m-(i-1)}}(t_k)\cdots\omega_2^{l_m}(t_k)}{|\xi||(\gamma'(t_k(\xi_0))\cdot\xi_0)||(\gamma(t_k(\xi_0))-x)|}, \tag{9}$$

where k is the number of intersection points of the plane $x+\xi^\perp$ with the given curve γ.

The entries of the principal symbol matrix $A_0(x,\xi)$ is obtained by fixing a lexicographic ordering of the indices of a symmetric m-tensor field. The proof of the above proposition follows by straightforward adaptation of the arguments given in [19, 25, 17].

Proposition 4.2. *The principal symbol matrix $A_0(x,\xi)$ is injective.*

Proof. For $(x,\xi) \in T^*(\mathbb{R}^3)\setminus 0$, without loss of generality, we choose a spherical coordinate system such that $\omega(\cdot)$ and $\omega_1(\cdot)$ are parallel to the plane $x+\xi^\perp$ and $\omega_2(\cdot)$ is in the direction of ξ.

By the Kirillov-Tuy condition, the hyperplane $x+\xi^\perp$ intersects the curve $\boldsymbol{\gamma}$ in at least $(m+1)$ points, say $t_1,\cdots,t_{m+1}$. Denote the collection of unit vectors from x to $x+\xi^\perp\cap\boldsymbol{\gamma}$ as

$$\mathbb{A} = \left\{\omega(t_k) := \frac{x-\boldsymbol{\gamma}_k}{|x-\boldsymbol{\gamma}_k|} : \boldsymbol{\gamma}_k = \boldsymbol{\gamma}(t_k), 1\le k\le m+1\right\}.$$

Now any two of the vectors in $\mathbb{A}$ are linearly independent by the Kirillov-Tuy condition. This in turn implies that for almost all points x, any two of the vectors in the collection

$$\mathbb{A}' = \left\{\omega_1(t_k) : 1\le k\le m+1\right\}$$

are also linearly independent. Denote the matrix $U_p = U_{\underbrace{1\dots1}_{p}\underbrace{2\dots2}_{m-p}}$, where $0\le p\le m$, whose columns are

$$\omega_1(t_k)^{\odot p}\odot\omega_2(t_k)^{\odot m-p} \text{ for } 1\le k\le m+1.$$

We write $A_0(x,\xi)$ as

$$A_0(x,\xi) = PP^t,$$

where P is defined in (10). In Lemma 4.4, we show that rank $(P) = (m+2)(m+1)/2$. Since P has real entries, $\operatorname{rank}(PP^t) = \operatorname{rank}(P)$. Therefore the principal symbol matrix $A_0(x,\xi)$ has full rank on $\Delta\setminus\Sigma$. $\square$

Lemma 4.3. *The rank of U_p is $p+1$.*

Proof. By the Kirillov-Tuy condition, we have a set consisting of at least $m+1$ pairwise linearly independent vectors $\{\omega(t_1),\cdots,\omega(t_{m+1})\}$. Now $\omega_1(t_1),\cdots,\omega_1(t_{m+1})$ are pairwise linearly independent and perpendicular to ξ. By the Kirillov-Tuy condition, the collection of vectors $\{\omega_1(t_1)^{\odot p},\cdots,\omega_1(t_{p+1})^{\odot p}\}$ has rank $p+1$. Then the rank of the matrix whose columns are $\omega_{\theta_1}(t_1)^{\odot p},\cdots,\omega_{\theta_1}(t_{m+1})^{\odot p}$ is at least $p+1$.

Finally, the rank of U_p is at least $p+1$ as well, since $\omega_2(t_k)$'s are in the direction of the nonzero vector ξ. We will be able to conclude that the rank is exactly $p+1$ as a consequence of the next lemma. $\square$

Let us denote the matrix P with column blocks $\{U_i\}, 0\le i\le m$:

$$P = \begin{pmatrix} U_m & U_{m-1} & \cdots & U_p & \cdots & U_0 \end{pmatrix} \tag{10}$$

Lemma 4.4. *The rank of P is $(m+2)(m+1)/2$.*

The proof of this is a straightforward consequence of the following lemma.

Lemma 4.5. *Consider an arbitrary U_l for $0\le l\le m$. Assume that the values of t_k corresponding to the linearly independent columns of U_l are $t_{k_1},\cdots,t_{k_{j+1}}$. Any column among these $l+1$ linearly independent columns cannot be written as a linear combination of the columns of the matrices U_p for $0\le p\le m, p\ne l$ and the remaining l linearly independent columns of the matrix U_l.*

Proof. Fix one of the linearly independent columns from U_l, say, $\omega(t_{k_1})^{\odot l}\odot\xi^{\odot m-l}$. Suppose there exists constants c_{pi}'s and d_j's such that

$$\omega_1(t_{k_1})^{\odot l}\odot\xi^{\odot m-l} = \sum_{p=0,p\ne l}^{m}\sum_{i=1}^{m+1} c_{pi}\omega_1(t_i)^{\odot p}\odot\xi^{\odot m-p} + \sum_{j=2}^{l+1} d_j\omega_1(t_{k_j})^{\odot l}\odot\xi^{\odot m-l}. \tag{11}$$

We can write $\omega_1(t_{k_i}) = \sum_{j=1}^{2} a_{ij}\omega_1(t_{k_j})$ for $i \geq 3$. Substituting this above, we have,

$$\begin{aligned}\omega_1(t_{k_1})^{\odot l} \odot \xi^{\odot m-l} = &\sum_{p=0,p\neq l}^{m}\left(c_{p1}\omega_1(t_{k_1})^{\odot p} + c_{p2}\partial_{\theta_1}\omega(t_{k_2})^{\odot p} + \sum_{i=3}^{m+1} c_{pi}\left(\sum_{j=1}^{2} a_{ij}\omega_1(t_{k_j})\right)^{\odot p}\right)\odot \xi^{\odot m-p}\\ &+\left(\sum_{j=3}^{l+1}\widetilde{d}_j a_{j1}^{l}\right)\omega_1(t_{k_1})^{\odot l}\odot\xi^{\odot m-l} + \left(d_2 + \sum_{j=3}^{l+1}\widetilde{d}_j a_{j2}^{l}\right)\partial_{\theta_1}\omega(t_{k_2})^{\odot l}\odot \xi^{\odot m-l}\\ &+\sum_{s=1}^{l-1}\sum_{j=3}^{l+1}\widetilde{d}_j a_{j1}^{l-s}a_{j2}^{s}\omega_1(t_{k_1})^{\odot l-s}\odot\omega_1(t_{k_2})^{\odot s}\odot\xi^{\odot m-l}\end{aligned}$$

This implies, for certain constants $c_{r_1r_2}$,

$$\begin{aligned}&\sum_{p=0,p\neq l}^{m}\sum_{r_1+r_2=p} c_{r_1r_2}\omega_1(t_{k_1})^{\odot r_1}\odot\omega_1(t_{k_2})^{\odot r_2}\odot\xi^{\odot m-p}\\ &+\left(\sum_{j=3}^{l+1}\widetilde{d}_j a_{j1}^{l} - 1\right)\omega_1(t_{k_1})^{\odot l}\odot\xi^{\odot m-l} + \left(d_2 + \sum_{j=3}^{l+1}\widetilde{d}_j a_{j2}^{l}\right)\omega_1(t_{k_2})^{\odot l}\odot\xi^{\odot m-l}\\ &+\sum_{s=1}^{l-1}\sum_{j=3}^{l+1}\widetilde{d}_j a_{j1}^{l-s}a_{j2}^{s}\partial_{\theta_1}\omega(t_1)^{\odot l-s}\odot\partial_{\theta_1}\omega(t_2)^{\odot s}\odot\xi^{\odot m-l} = 0.\end{aligned}$$

The vectors $\{\omega_1(t_{k_1}), \omega_1(t_{k_2}), \xi\}$ are linearly independent. Therefore the collection of tensors $\{\omega_1(t_{k_1})^{\odot k_1}\odot \omega_1(t_{k_2})^{\odot k_2}\odot\xi^{\odot k_3} : k_1+k_2+k_3 = m\}$ is also linearly independent. Thus

$$\begin{aligned} c_{r_1r_2} &= 0\\ \sum_{j=3}^{l+1}\widetilde{d}_j a_j^{l} - 1 &= 0\\ d_2 + \sum_{j=3}^{l+1}\widetilde{d}_j a_{j2}^{l} &= 0\\ \sum_{j=3}^{l+1}\widetilde{d}_j a_{j1}^{l-s}a_{j2}^{s} &= 0 \quad \text{for } 1 \leq s \leq l-1.\end{aligned} \tag{12}$$

Since a_{j1} and a_{j2} are non-zero, the last of the equations in (12) can be written as

$$AX = 0,$$

where

$$A = \begin{pmatrix} a_{31}^{l-2} & \cdots & a_{l+11}^{l-2}\\ a_{31}^{l-3}a_{32} & \cdots & a_{l+11}^{l-3}a_{l+12}\\ \vdots & \ddots & \vdots\\ a_{32}^{l-2} & \cdots & a_{l+12}^{l-2}\end{pmatrix}$$
$$\text{and}\quad X = (\widetilde{d}_3, \widetilde{d}_4, \cdots, \widetilde{d}_{l+1}).$$

Let $b_j = (a_{j1}, a_{j2})$ for $3 \leq j \leq l+1$. Since any two vectors from $\{\omega_1(t_{k_j}) : 3 \leq j \leq l+1\}$ are linearly independent, any two vectors from the set $\{b_j : 3 \leq j \leq l+1\}$ are also linearly independent and the columns of A are $\{b_j^{\odot l-2}, 3 \leq j \leq l+1\}$. Therefore by the Kirillov-Tuy condition for $m = l-2$, we have that the matrix A has full rank. Hence $\{\widetilde{d}_j = 0, 3 \leq j \leq l+1\}$. However, this contradicts the second equation in (12). This completes the proof. □

Now going back to the proof of Lemma 4.3, we have that the rank of U_p is exactly $p+1$ as well.

Remark 4.6. In the general case of fixing a spherical coordinate system independent of the plane $x+\xi^{\perp}$, the arguments would follow similarly as above, except that, one would need to consider linear combinations of the components $\mathcal{T}_i$ of the TRT $\mathcal{T}$ in the proofs above.

5. Microlocal inversion

In this section, we will give a relative left parametrix for the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma$. This will complete the proof of Theorem 2.3.

Proof of Theorem 2.3. Now that ellipticity of $A_0(x,\xi)$ is shown, the construction of the relative left parametrix follows the arguments of [25, 17]. For the sake of completeness, we sketch the proof.

Since $A_0(x,\xi)$ is a symmetric matrix of order $(m+1)(m+2)/2$, we diagonalize $A_0(x,\xi)$ by an orthogonal matrix $\mathcal{O}$ such that

$$A_0(x,\xi)=\mathcal{O}D\mathcal{O}^t,$$

where D is the diagonal matrix consisting eigenvalues of A_0 and $\mathcal{O}$ is an orthogonal matrix whose columns are eigenvectors corresponding to the eigenvalues of A_0. Since A_0 has full rank, all diagonal entries in D are non-zero. Let

$$B_0(x,\xi)=\mathcal{O}D^-\mathcal{O}^t$$

where D^- is a matrix obtained from D by taking the reciprocal of the diagonal elements. We have

$$B_0(x,\xi)A_0(x,\xi)=\mathrm{Id}.$$

The entries of $B_0(x,\xi)$ belong to the symbol of an $I^{p,l}(\Delta,\Lambda)$ class, since the possible singularities of $\mathcal{O}$ and D^- are only on Σ. Define the matrix b_0 as

$$b_0=\begin{cases}B_0 \text{ if } (x,\xi)\in\Xi_0,\\ 0 \text{ otherwise.}\end{cases} \tag{13}$$

and $\mathcal{B}_0$ be the operator with symbol matrix $b_0(x,\xi)$.

Now the operator $\mathcal{T}_\gamma^*\mathcal{T}_\gamma\in I^{-1,0}(\Delta,\Lambda)$, and since the principal symbol of the composition $\mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma$ on Δ away from the intersection $\Delta\cap\Lambda$ is the product of the respective principal symbols by [3], which by construction is the identity on Δ away from $\Delta\cap\Lambda$, we have that $\mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma\in I^{-\frac{1}{2},\frac{1}{2}}(\Delta,\Lambda)$.

Define $T_1=\mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma-\mathrm{Id}$. By construction the pricipal symbol of T_1 is 0. We now use the exact sequence [13]

$$0\to I^{p,l-1}(\Delta,\Lambda)+I^{p-1,l}(\Delta,\Lambda)\to I^{p,l}(\Delta,\Lambda)\xrightarrow{\sigma_0}S^{p,l}(\Delta,\Sigma)\to 0.$$

to decompose T_1 as $T_1=T_{11}+T_{12}$ where $T_{11}\in I^{-\frac{3}{2},\frac{1}{2}}$ and $T_{12}\in I^{-\frac{1}{2},-\frac{1}{2}}$.

Since A_0 has full rank, we can find two matrices t_{11} and t_{12} such that the principal symbol $\sigma_0(T_{1j})=t_{1j}A_0$ for $j=1,2$.

Let $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ be the operators having symbol matrices $-t_{11}$ and $-t_{12}$ respectively. For $\mathcal{B}_1=\mathcal{B}_{11}+\mathcal{B}_{12}$, define $T_2=(\mathcal{B}_0+\mathcal{B}_1)\mathcal{T}_\gamma^*\mathcal{T}_\gamma-\mathrm{Id}$. We have

$$\begin{aligned}T_2&=(\mathcal{B}_0+\mathcal{B}_1)\mathcal{T}_\gamma^*\mathcal{T}_\gamma-\mathrm{Id}\\&=\mathcal{B}_{11}\mathcal{T}_\gamma^*\mathcal{T}_\gamma+\mathcal{B}_{12}\mathcal{T}_\gamma^*\mathcal{T}_\gamma+\mathcal{B}_0\mathcal{T}_\gamma^*\mathcal{T}_\gamma-\mathrm{Id}\\&=\underbrace{\mathcal{B}_{11}\mathcal{T}_\gamma^*\mathcal{T}_\gamma+T_{11}}_{K_1}+\underbrace{\mathcal{B}_{12}\mathcal{T}_\gamma^*\mathcal{T}_\gamma+T_{12}}_{K_2}.\end{aligned}$$

In the above expression $K_1\in I^{-\frac{3}{2},\frac{1}{2}}$ and $K_2\in I^{-\frac{1}{2},-\frac{1}{2}}$. Also, by construction, $\sigma_0(K_1)=0$ and $\sigma_0(K_2)=0$ because $\sigma_0(\mathcal{B}_{11}\mathcal{T}_\gamma^*\mathcal{T}_\gamma)=-\sigma_0(T_{11})$ and $\sigma_0(\mathcal{B}_{12}\mathcal{T}_\gamma^*\mathcal{T}_\gamma)=-\sigma_0(T_{12})$. Therefore we can again use symbol calculus to decompose K_1 and K_2 as follows:

$$\begin{aligned}K_1&=K_{11}+K_{12}, &&\text{with } K_{11}\in I^{-\frac{5}{2},\frac{1}{2}},K_{12}\in I^{-\frac{3}{2},-\frac{1}{2}}\\K_2&=K_{21}+K_{22}, &&\text{with } K_{21}\in I^{-\frac{3}{2},-\frac{1}{2}},K_{22}\in I^{-\frac{1}{2},-\frac{3}{2}}.\end{aligned}$$

Putting this in T_2, we get

$$T_2 = \underbrace{K_{11}}_{T_{20}} + \underbrace{K_{12} + K_{21}}_{T_{21}} + \underbrace{K_{22}}_{T_{22}}$$

where $T_{20} \in I^{-\frac{5}{2},\frac{1}{2}}$, $T_{21} \in I^{-\frac{3}{2},-\frac{1}{2}}$, $T_{22} \in I^{-\frac{1}{2},-\frac{3}{2}}$. Therefore

$$T_2 \in \sum_{j=0}^{2} I^{-\frac{1}{2}-2+j,\frac{1}{2}-j}.$$

Proceeding recursively, we get a sequence of operators

$$T_N \in \sum_{j=0}^{N} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j}.$$

We can break this sum as follows:

$$T_N \in \sum_{j=0}^{[\frac{N}{2}]} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j} + \sum_{j=[\frac{N}{2}]+1}^{N} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j}.$$

In the first sum $-\frac{1}{2} - N + j \leq -\frac{1}{2} - N + \left[\frac{N}{2}\right]$ and $\frac{1}{2} - j \leq \frac{1}{2}$. Similarly in the second sum, $-\frac{1}{2} - N + j \leq -\frac{1}{2}$ and $\frac{1}{2} - j \leq -\frac{1}{2} - \left[\frac{N}{2}\right]$. Now we use $I^{p,l} \subset I^{p',l'}$ for $p \leq p'$, $l \leq l'$ to get

$$\sum_{j=0}^{[\frac{N}{2}]} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j} \in I^{-\frac{1}{2}-N+\left[\frac{N}{2}\right],\frac{1}{2}} \text{ and } \sum_{j=[\frac{N}{2}]+1}^{N} I^{-\frac{1}{2}-N+j,\frac{1}{2}-j} \in I^{-\frac{1}{2},-\frac{1}{2}-\left[\frac{N}{2}\right]}.$$

In the limit $N \to \infty$, the first term in the above expression is a smoothing term by the property that $\cap_p I^{p,l}(\Delta,\Lambda) \subset \mathcal{C}^\infty$ and the second term is an operator $\mathcal{A}$ in $I^{-\frac{1}{2}}(\Lambda)$ by the property $\cap_l I^{p,l}(\Delta,\Lambda) \subset I^p(\Lambda)$. Finally, we define $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \cdots$ and from the construction above, we get,

$$\mathcal{B}\mathcal{T}_\gamma^*\mathcal{T}_\gamma(f) = f + \mathcal{A}f + \mathcal{C}^\infty.$$

This completes the proof of the Theorem 2.3. □

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