UNIQUE CONTINUATION RESULTS FOR CERTAIN GENERALIZED RAY TRANSFORMS OF SYMMETRIC TENSOR FIELDS

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Abstract. Let I_m denote the Euclidean ray transform acting on compactly supported symmetric m-tensor field distributions f, and I_m^* be its formal L^2 adjoint. We study a unique continuation result for the normal operator $N_m = I_m^* I_m$. More precisely, we show that if N_m vanishes to infinite order at a point x_0 and if the Saint-Venant operator W acting on f vanishes on an open set containing x_0, then f is a potential tensor field. This generalizes two recent works of Ilmavirta and Mönkkönen who proved such unique continuation results for the ray transform of functions and vector fields/1-forms. One of the main contributions of this work is identifying the Saint-Venant operator acting on higher order tensor fields as the right generalization of the exterior derivative operator acting on 1-forms, which makes unique continuation results for ray transforms of higher order tensor fields possible. In the second half of the paper, we prove analogous unique continuation results for momentum ray and transverse ray transforms.

1. Introduction

The purpose of this paper is to prove unique continuation properties (UCP) for three Euclidean ray transforms of symmetric m-tensor fields; the (usual) ray transform, momentum ray transform and transverse ray transform. Roughly speaking, we show the following: Let f be a compactly supported m-tensor field distribution and U be a non-empty open subset of $\mathbb{R}^n$ for $n \geq 2$.

(1) If the ray transform of f vanishes u and if the Saint-Venant operator acting on f vanishes on the same open set, then f is a potential tensor field.
(2) If certain momentum ray transforms of f vanish on a set of lines passing through U and if the generalized Saint-Venant operator acting on f vanishes on the same open set, then f is a generalized potential tensor field.
(3) Let $n \geq 3$. If the transverse ray transform of f vanishes on a set of lines passing through U and if f vanishes on the same open set, then f vanishes identically.

We actually prove stronger versions of some of the the statements mentioned above; see the precise statements of the theorems in the concluding paragraphs of Section 2.

The study of the three transforms on symmetric tensor fields is motivated by applications in several applied fields. The investigation of ray transform of symmetric 2-tensor fields is motivated by applications in travel-time tomography [20, 23] and that of symmetric 4-tensor fields in elasticity [20]. The study of momentum ray transforms was introduced by Sharaftudinov [20] and a more detailed investigation of this transform was undertaken in [1, 10, 11, 16]. Analysis of such transforms appeared recently in the solution of a Calderón-type inverse problem for polyharmonic operators; see [2]. Transverse ray transform of symmetric tensor fields appear in the study of polarization tomography [20, 17, 15] and X-ray diffraction strain tomography [14, 4].

We note that the recovery of a symmetric m-tensor field f from the knowledge of its ray transform $I_m f$ is an over-determined problem in dimensions $n \geq 3$. However, the recovery of f given the normal operator $N_m f = I_m^* I_m f$, when viewed as a convolution operator; see (2.16), is a formally determined inverse problem, where I_m^* is the formal L^2 adjoint. Furthermore, we study a partial data problem, that is, the recovery of f, from the knowledge of $N_m f$ and a component of f given in a fixed open subset of $\mathbb{R}^n$. We prove a unique continuation result for this as well as for two other transforms; the momentum ray and transverse ray transforms. The motivation for a result of this kind for the ray transform comes from its connection to the fractional Laplacian operator. The inversion formula for the recovery of a function (for instance) from its corresponding normal operator is given by the following formula: $f = C(-\Delta)^{1/2} I_0^* I_0 f$, where C is a

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constant that depends only on dimension. Unique continuation results for the fractional operators have a long history and go back to the works of Riesz [18] and Kotake-Narasimhan [9]. A unique continuation result for a fractional Schrödinger equation with rough potentials was done in [19]. In the context of a Calderón-type inverse problem involving the fractional Laplacian, a unique continuation result was employed to prove uniqueness in [5].

Unique continuation results for the ray transform of functions, the d-plane transform and the Radon transform were initiated in the paper [6]. Later this was extended to a unique continuation result for the Doppler transform, which deals with the ray transform of a vector field or equivalently a 1-form in [7]. One added difficulty in dealing with the Doppler transform or ray transform of higher order symmetric tensor fields is that the ray transform has an infinite dimensional kernel. Therefore unique recovery of the full symmetric tensor field from its ray transform is not possible. Going back to the paper [7], roughly speaking, the main result of the paper reads as follows: Let f be a compactly supported vector field/1-form and suppose $\mathrm{d}f = 0$ on an non-empty open set U, where $\mathrm{d}f$ is the exterior derivative of the 1-form f, and if the Doppler transform of f vanishes along all lines intersecting U, then $\mathrm{d}f \equiv 0$ in $\mathbb{R}^n$. The results of our paper can be viewed as a generalization of this work. The approach of [7] is to reduce the unique continuation result for the Doppler transform to that of the scalar ray transform of each component of $\mathrm{d}f$. We follow their idea of reducing to a unique continuation result for a scalar function for the symmetric tensor field case, however, our approach as well as the technique of proof are different. Our main contribution is in identifying the right analogue of the exterior derivative operator to higher order symmetric tensor fields case, which turns out to be the Saint-Venant operator, to prove the unique continuation results for ray transform of higher order symmetric tensor fields. We also prove unique continuation results for momentum ray transforms as well as for transverse ray transform of symmetric tensor fields. To prove unique continuation results for the momentum ray transform, we consider the generalized Saint-Venant operator introduced by Sharafutdinov [20]. In fact, we define an equivalent version of the generalized Saint-Venant operator from [20] suitable for our purposes to prove our result.

The article is organized as follows. In Section 2, we give the requisite preliminaries and give the statement of the main results. Readers familiar with the integral geometry literature may choose to skip the parts of the section where we fix the notation required to give the statements of the theorems. Instead, they may go directly to the results near the end of Section 2 and refer back to the preliminary material as and when required. Sections 3, 4 and 5 give the proofs of the unique continuation results for ray transform, momentum ray transform and transverse ray transform, respectively.

2. Preliminaries and statements of the main results

To state the main results of this work, we begin by defining the operators that will be used throughout the article. Most of these are standard in integral geometry literature and the reference is the book by Sharafutdinov [20]. For the purpose of fixing the notation, we give them here.

2.1. Definitions of some operators.

We let $T^m = T^m\mathbb{R}^n$ denote the complex vector space of $\mathbb{R}$-multilinear functions from $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m \text{ times}} \to \mathbb{C}$. Let $e_1, \cdots, e_n$ be the standard basis for $\mathbb{R}^n$. Given an element $u \in T^m$, we let $u_{i_1\cdots i_m} = u(e_{i_1}, \cdots, e_{i_m})$. These are the components of the tensor T^m.

Given $u \in T^m$ and $v \in T^k$, the tensor product $u \otimes v \in T^{m+k}$ is defined by

$$(u \otimes v)(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+k}) = u(x_1, \cdots, x_m)v(x_{m+1}, \cdots, x_{m+k}).$$

By $S^m = S^m\mathbb{R}^n$, we mean the subspace of T^m that are symmetric in all its m arguments. More precisely, u is an element of S^m if

$$u_{i_1\cdots i_m} = u_{i_{\pi(1)}\cdots i_{\pi(m)}}$$

for any $\pi \in \Pi_m$ – the group of permutations of the set $\{1, \cdots, m\}$.

Let $\sigma : T^m \to S^m$ be the symmetrization operator defined as follows:

$$\sigma u(e_1, \cdots, e_m) = \frac{1}{m!} \sum_{\pi \in \Pi_m} u(e_{\pi(1)}, \cdots, e_{\pi(m)}).$$

The symmetrized tensor product of two tensors will be denoted by $\odot$ instead of $\otimes$. That is, given $u \in T^m$ and $v \in T^k$,

$$(u \odot v)\,(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+k}) = \frac{1}{(m+k)!} \sum_{\pi \in \Pi_{m+k}} u(x_{\pi(1)}, \cdots, x_{\pi(m)}) v(x_{i_{\pi(m+1)}}, \cdots, x_{i_{\pi(m+k)}}).$$

Given indices $i_1, \cdots, i_m$ the operator of partial symmetrization with respect to the indices $i_1, \ldots, i_p$, where $p < m$, of a tensor $u \in T^m$ is given by

$$\sigma(i_1 \ldots i_p) u_{i_1 \ldots i_m} = \frac{1}{p!} \sum_{\pi \in \Pi_p} u_{i_{\pi(1)} \ldots i_{\pi(p)} i_{p+1} \ldots i_m},$$

where Π_p denotes the group of permutations of the set $\{1, \ldots, p\}$.

We next define symmetric tensor fields. If $\mathcal{A} \subset \mathcal{D}'(\mathbb{R}^n)$, the space of $\mathcal{A}$-valued symmetric tensor field distributions of $\mathbb{R}^n$ is defined by $\mathcal{A}(\mathbb{R}^n; S^m) = \mathcal{A} \otimes_{\mathbb{C}} S^m$. Denote by $C^\infty(\mathbb{R}^n; S^m)$, $\mathcal{S}(\mathbb{R}^n; S^m)$, $C_c^\infty(\mathbb{R}^n; S^m)$, $\mathcal{D}'(\mathbb{R}^n; S^m)$, $\mathcal{S}'(\mathbb{R}^n; S^m)$ and $\mathcal{E}'(\mathbb{R}^n; S^m)$ the space of symmetric m-tensor fields in $\mathbb{R}^n$ whose components are smooth, Schwartz class, smooth and compactly supported functions, tensor field, tempered tensor field, and compactly supported tensor field distributions, respectively. An analogous definition is valid when S^m above is replaced by T^m.

The family of oriented lines in $\mathbb{R}^n$ is parameterized by

$$T\mathbb{S}^{n-1} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : \langle x, \xi \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard dot product in $\mathbb{R}^n$.

The alternation operator α with respect to two indices i_1, i_2 is defined by

$$\alpha(i_1\, i_2) u_{i_1 i_2 j_1 \ldots j_p} = \frac{1}{2}(u_{i_1 i_2 j_1 \ldots j_p} - u_{i_2 i_1 j_1 \ldots j_p}).$$

For $u \in S^k$, we denote by $i_u : S^m \to S^{m+k}$ the operator of symmetric multiplication by u and by $j_u : S^{m+k} \to S^m$ the operator dual to i_u. These are given by

$$(i_u f)_{i_1 \ldots i_{m+k}} = \sigma(i_1 \ldots i_{m+k}) u_{i_1 \ldots i_k} f_{i_{k+1} \ldots i_{k+m}} \tag{2.1}$$

$$(j_u g)_{i_1 \ldots i_m} = g_{i_1 \ldots i_{m+k}} u^{i_{m+1} \ldots i_{m+k}}. \tag{2.2}$$

For the case in which u is the Euclidean metric tensor, we denote i_u and j_u by i and j respectively. In (2.2) and henceforth, we use the Einstein summation convention, that when the indices are repeated, summation in each of the repeating index varying from 1 up to the dimension n is assumed.

Next we define two important first order differential operators. The operator of inner differentiation or symmetrized derivative is denoted as $\mathrm{d} : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^{m+1})$ given by

$$(\mathrm{d}f)_{i_1 \ldots i_{m+1}} = \sigma(i_1 \ldots i_{m+1}) \frac{\partial f_{i_1 \ldots i_m}}{\partial x_{i_{m+1}}}.$$

The divergence operator $\delta : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^{m-1})$ is defined by

$$(\delta f)_{i_1 \ldots i_{m-1}} = \frac{\partial f_{i_1 \ldots i_m}}{\partial x^{i_m}}. \tag{2.3}$$

The operators d and $-\delta$ are formally dual to each other with respect to L^2 inner product.

$$\langle u, v \rangle = \int u_{i_1 \cdots i_m} \overline{v}^{i_1 \cdots i_m} \, \mathrm{d}x.$$

Note that the above definitions make sense for compactly supported tensor field distributions as well.

We now recall the solenoidal-potential decomposition of compactly supported symmetric tensor field distributions [20]. Let $n \geq 2$. For $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$, there exist uniquely determined fields ${}^s f \in \mathcal{S}'(\mathbb{R}^n; S^m)$ and $v \in \mathcal{S}'(\mathbb{R}^n; S^{m-1})$ tending to 0 at ∞ and satisfying

$$f = {}^s f + \mathrm{d}v, \quad \delta^s f = 0.$$

The fields ${}^s f$ and $\mathrm{d}v$ are called the solenoidal and potential components of f, respectively. The fields ${}^s f$ and v are smooth outside $\operatorname{supp} f$ and satisfy the estimates

$$|{}^s f(x)| \leq C(1 + |x|)^{1-n}, |v(x)| \leq C(1 + |x|)^{2-n}, |\mathrm{d}v(x)| \leq C(1 + |x|)^{1-n}.$$

Finally, we define the Saint-Venant operator W and the generalized Saint-Venant operator W^k for $0 \leq k \leq m$.

The Saint-Venant operator $W : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^m \otimes S^m)$ is the differential operator of order m defined by

$$(Wf)_{i_1 \dots i_m j_1 \dots j_m} = \sigma(i_1 \dots i_m)\sigma(j_1 \dots j_m) \sum_{p=0}^{m} (-1)^p \binom{m}{p} \frac{\partial^m f_{i_1 \dots i_{m-p} j_1 \dots j_p}}{\partial x_{j_{p+1}} \dots \partial x_{j_m} \partial x_{i_{m-p+1}}, \dots \partial x_{i_m}}, \tag{2.4}$$

where $S^m \otimes S^m$ denotes the set of tensors symmetric with respect to the group of first and last m indices.

For our purposes, we will be using an equivalent formulation of the Saint-Venant operator using the operator $R : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; T^{2m})$ defined as follows: For $f \in C^\infty(\mathbb{R}^n; S^m)$,

$$(Rf)_{i_1 j_1 \dots i_m j_m} = \alpha(i_1 j_1) \dots \alpha(i_m j_m) \frac{\partial^m f_{i_1 \dots i_m}}{\partial x_{j_1} \dots \partial x_{j_m}}. \tag{2.5}$$

The tensor field R is skew-symmetric with respect to each pair of indices $(i_1, j_1), \dots (i_m, j_m)$, and symmetric with respect to these pairs. The operators R and W are equivalent. More precisely,

$$(Wf)_{i_1 \dots i_m j_1 \dots j_m} = 2^m \sigma(i_1 \dots i_m)\sigma(j_1 \dots j_m)(Rf)_{i_1 j_1 \dots i_m j_m} \tag{2.6}$$

$$(Rf)_{i_1 j_1 \dots i_m j_m} = \frac{1}{(m+1)} \alpha(i_1 j_1) \dots \alpha(i_m j_m)(Wf)_{i_1 \dots i_m j_1 \dots j_m}. \tag{2.7}$$

We remark that in [20], these formulas have two minor typos; the factor 2^m is missing in (2.6) and the constant in (2.7) is incorrectly written as $(m+1)$ on the right hand side.

We also need the following generalization [20] of the Saint-Venant operator.

For $m \geq 0$ and $0 \leq k \leq m$, the generalized Saint-Venant operator $W^k : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^{m-k} \otimes S^m)$ is defined as

$$\begin{aligned}(W^k f)_{p_1 \dots p_{m-k} q_1 \dots q_{m-k} i_1 \dots i_k} =& \sigma(p_1, \dots, p_{m-k})\sigma(q_1, \dots, q_{m-k}, i_1 \dots i_k) \\ & \sum_{l=0}^{m-k} (-1)^l \binom{m-k}{l} \frac{\partial^{m-k} f^{i_1 \dots i_k}_{p_1 \dots p_{m-k-l} q_1 \dots q_l}}{\partial x^{p_{m-k-l+1}} \dots \partial x^{p_{m-k}} \partial x^{q_{l+1}} \dots \partial x^{q_{m-k}}},\end{aligned} \tag{2.8}$$

where we adopt the notation from [16] and by $f^{i_1 \dots i_k}_{p_1 \dots p_{m-k}}$, we mean a tensor field of order $m-k$ with the indices on the top fixed ($i_1 \dots i_k$ here). Note that for $k = 0$, (2.8) agrees with (2.4), and for $k = m$, $W^m = I$ – the identity operator.

Next we define a generalization of the operator R as follows. For $f \in C^\infty(\mathbb{R}^n; S^m)$,

$$\begin{aligned}(R^k f)_{p_1 q_1 \dots p_{m-k} q_{m-k} i_1 \dots i_k} &= \alpha(p_1 q_1) \dots \alpha(p_{m-k} q_{m-k}) \frac{\partial^{m-k} f^{i_1 \dots i_k}_{p_1 \dots p_{m-k}}}{\partial x_{q_1} \dots \partial x_{q_{m-k}}} \\ &= R(f^{i_1 \dots i_k})_{p_1 q_1 \dots p_{m-k} q_{m-k}}.\end{aligned} \tag{2.9}$$

As with the equivalence of W and R, we show the equivalence of W^k and R^k in Section 4.

Finally, note that the operators W, R, W^k and R^k are well-defined for tensor field distributions as well.

2.2. **Ray, momentum and transverse ray transforms.** We now define the ray transform, momentum and transverse ray transforms whose unique continuation properties we study in this paper. We initially define these transforms on the space of smooth compactly supported symmetric tensor fields. These will be extended to compactly supported tensor field distributions later.

The ray transform I is the bounded linear operator

$$I_m : C_c^\infty(\mathbb{R}^n; S^m) \to C_c^\infty(T\mathbb{S}^{n-1})$$

defined as follows:

$$I_m f(x, \xi) = \int_{\mathbb{R}} f_{i_1 \cdots i_m}(x + t\xi)\xi^{i_1} \cdots \xi^{i_m} \, \mathrm{d}t = \int_{\mathbb{R}} \langle f(x + t\xi), \xi^m \rangle \, \mathrm{d}t \tag{2.10}$$

This can be naturally extended to points $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ using the same definition. We will denote the extended operator by $J_m : C_c^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$:

$$J_m f(x,\xi) = \int f_{i_1\cdots i_m}(x+t\xi)\xi^{i_1}\cdots\xi^{i_m}\,\mathrm{d}t.$$

In fact, the operators I_m and J_m are equivalent. Restricting $J_m f$ to points on $T\mathbb{S}^{n-1}$ determines $I_m f$. For the other way, we use the following homogeneity properties:

$$J_m(x,r\xi) = \frac{r^m}{|r|}(J_m^k f)(x,\xi) \quad \text{for} \quad r \neq 0,$$
$$J_m(x+s\xi,\xi) = J_m(x,\xi).$$

From this we have

$$J_m f(x,\xi) = |\xi|^{m-1} I_m f\left(x - \frac{\langle x,\xi\rangle\xi}{|\xi|^2}, \frac{\xi}{|\xi|}\right) \text{ for } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

The main reason for working with J_m instead of I_m is that the partial derivatives $\dfrac{\partial(Jf)}{\partial x_i}$ and $\dfrac{\partial(Jf)}{\partial \xi_i}$ are well-defined for all $1 \leq i \leq n$.

The momentum ray transforms are the bounded linear operators, $I_m^k : C_c^\infty(\mathbb{R}^n; S^m) \to C_c^\infty(T\mathbb{S}^{n-1})$, defined for each $k \geq 0$ as

$$(I_m^k f)(x,\xi) = \int_{-\infty}^{\infty} t^k f_{i_1\cdots i_m}(x+t\xi)\xi^{i_1}\cdots\xi^{i_m}\,\mathrm{d}t. \tag{2.11}$$

The operator I_m^0 is of course the ray transform I defined above.

Similar to the case of ray transform, the momentum ray transforms can be extended for points $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. The extended operators will be denoted by

$$J_m^k : C_c^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}) \tag{2.12}$$

using the same definition. The operators J_m^k satisfy the following [10]:

$$(J_m^k f)(x,r\xi) = \frac{r^{m-k}}{|r|}(J_m^k f)(x,\xi) \quad \text{for} \quad r \neq 0 \tag{2.13}$$

$$(J_m^k f)(x+s\xi,\xi) = \sum_{l=0}^{k} \binom{k}{l}(-s)^{k-l}(J_m^l f)(x,\xi) \quad \text{for} \quad s \in \mathbb{R}. \tag{2.14}$$

The data $(I_m^0 f, I_m^1 f, \dots, I_m^k f)$ and $(J_m^0 f, J_m^1 f, \dots, J_m^k f)$ for $0 \leq k \leq m$ are equivalent ([10, Equations 2.5, 2.6]). As with the case of the ray transform, it is convenient to work with the operators J_m^k because the partial derivatives $\dfrac{\partial(J_m^k f)}{\partial x^i}$ and $\dfrac{\partial(J_m^k f)}{\partial \xi^i}$ are well-defined for all $1 \leq i \leq n$.

For $f \in C_c^\infty(\mathbb{R}^n; S^m)$, we define the transverse ray transform. Let

$$T\mathbb{S}^{n-1} \oplus T\mathbb{S}^{n-1} = \{(\omega,x,y) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n : \omega\cdot x = 0, \omega\cdot y = 0\}$$

be the Whitney sum.

The transverse ray transform $\mathcal{T} : C_c^\infty(\mathbb{R}^n; S^m) \to C^\infty(T\mathbb{S}^{n-1} \oplus T\mathbb{S}^{n-1})$ is the bounded linear map defined by

$$\mathcal{T} f(\omega,x,y) = \int_{\mathbb{R}} f_{i_1\cdots i_m}(x+t\omega)y^{i_1}\cdots y^{i_m}\,\mathrm{d}t. \tag{2.15}$$

2.3. **Normal operators.** Next we extend the definitions of the ray transforms to compactly supported tensor field distributions. We also define the corresponding normal operators.

For the case of ray transform (2.10), the definition can be extended to compactly supported tensor field distributions as done in [20]:

For $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$, we define $I_m f \in \mathcal{E}'(T\mathbb{S}^{n-1})$ as

$$\langle I_m f, \varphi\rangle = \langle f, I_m^* \varphi\rangle \text{ for } \varphi \in C^\infty(T\mathbb{S}^{n-1}),$$

where

$$(I_m^*)_{i_1\cdots i_m}\varphi(x)=\int\limits_{\mathbb{S}^{n-1}}\xi_{i_1}\cdots\xi_{i_m}\varphi(x-\langle x,\xi\rangle\xi,\xi)\,\mathrm{dS}_\xi.$$

Here and henceforth, dS_ξ is the Euclidean surface measure on the unit sphere.

Similarly, if we work with J_m, we can define $J_m f$ for $f\in\mathcal{E}'(\mathbb{R}^n;S^m)$ using the following formal L^2 adjoint:

$$(J_m^*)_{i_1\cdots i_m}\varphi(x)=\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\mathbb{R}}\xi_{i_1}\cdots\xi_{i_m}\varphi(x-t\xi,\xi)\mathrm{d}t\mathrm{dS}_\xi \text{ for } \varphi\in C_c^\infty(\mathbb{R}^n\times\mathbb{S}^{n-1}).$$

The normal operator $N_m f=I_m^*I_m f$ has the following integral representation [20].

$$(N_m f)_{i_1\cdots i_m}=2f_{j_1\cdots j_m}*\frac{\left(x^{\odot 2m}\right)_{j_1\cdots j_m i_1\cdots i_m}}{\|x\|^{2m+n-1}}. \tag{2.16}$$

This representation makes sense for $f\in\mathcal{E}'(\mathbb{R}^n;S^m)$ as the convolution of a compactly supported distribution and a tempered distribution.

Next we define momentum ray transform of compactly supported tensor distributions. This was studied recently in the context of an inverse problem for polyharmonic operators in [2]. We work in a slightly different context here and for this reason, we give the details.

Let us first derive a representation for the formal L^2 adjoint $\left(I_m^k\right)^*$ of I_m^k. Consider for $f\in C_c^\infty(\mathbb{R}^n)$ and $g\in C^\infty(T\mathbb{S}^{n-1})$

$$\begin{aligned}\langle I_m^k f,g\rangle_{T\mathbb{S}^{n-1}}&=\langle f,(I_m^k)^*g\rangle_{\mathbb{R}^n}\\&=\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp}(I_m^k f)(x,\xi)\,g(x,\xi)\,\mathrm{d}x\,\mathrm{dS}_\xi\\&=\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\xi^\perp}\int\limits_{-\infty}^{\infty}t^k\,\langle f(x+t\xi),\xi^m\rangle\,\mathrm{d}t\,g(x,\xi)\,\mathrm{d}x\,\mathrm{dS}_\xi\\&=\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\mathbb{R}^n}\langle z,\xi\rangle^k\,\langle f(z),\xi^m\rangle\,g(z-\langle z,\xi\rangle\xi,\xi)\,\mathrm{d}z\,\mathrm{dS}_\xi,\end{aligned}$$

where we employed the change of variables $x+t\xi=z$ for $x\in\xi^\perp$ and $t\in\mathbb{R}$ for each fixed $\xi\in\mathbb{S}^{n-1}$. Now interchanging the order of integration

$$\langle f,(I_m^k)^*g\rangle_{\mathbb{R}^n}=\int\limits_{\mathbb{R}^n}f_{i_1\dots i_m}(z)\left\{\int\limits_{\mathbb{S}^{n-1}}\langle z,\xi\rangle^k\xi_{i_1}\dots\xi_{i_m}\,g(z-\langle z,\xi\rangle\xi,\xi)\,\mathrm{d}\xi\right\}\mathrm{d}z.$$

Thus the formal L^2-adjoint of I_m^k, $(I_m^k)^*:C^\infty(T\mathbb{S}^{n-1})\to C^\infty(\mathbb{R}^n;S^m)$ is given by the expression

$$(I_m^k)^*g_{i_1\dots i_m}(x)=\int\limits_{\mathbb{S}^{n-1}}\langle x,\xi\rangle^k\,\xi_{i_1}\cdots\xi_{i_m}\,g(x-\langle x,\xi\rangle\xi,\xi)\,\mathrm{dS}_\xi. \tag{2.17}$$

Using this we can extend momentum ray transforms for compactly supported tensor field distributions as follows. $I_m^k:\mathcal{E}'(\mathbb{R}^n;S^m)\to\mathcal{E}'(T\mathbb{S}^{n-1})$ given by

$$\langle I_m^k f,g\rangle=\langle f,(I_m^k)^*g\rangle \tag{2.18}$$

for $f\in\mathcal{E}'(\mathbb{R}^n;S^m)$ and $g\in C^\infty(T\mathbb{S}^{n-1})$.

Similarly, if we work with J_m^k, then for $f\in\mathcal{E}'(\mathbb{R}^n;S^m)$, $J_m^k f$ can be defined as follows:

$$\langle J_m^k f,g\rangle=\langle f,(J_m^k)^*g\rangle$$

where

$$(J_m^k)^*_{i_1\cdots i_m}g(x)=\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\mathbb{R}}t^k g(x-t\xi,\xi)\xi^{i_1}\cdots\xi^{i_m}\,\mathrm{d}t\,\mathrm{dS}_\xi\quad\text{ for }\quad g\in C_c^\infty(\mathbb{R}^n\times\mathbb{S}^{n-1}).$$

We next study normal operators of momentum ray transforms. Let us denote $N^k_m = (I^k_m)^* I^k_m : C^\infty_c(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^m)$ be the normal operator of the k^{th} momentum ray transform of a symmetric m-tensor field. By (2.17),

$$\begin{aligned}(N^k_m\, f)_{i_1\dots i_m}(x) &= (I^k_m)^*_{i_1\cdots i_m}\, I^k_m\, f\,(x)\\ &= \int\limits_{\mathbb{S}^{n-1}} \langle x,\xi\rangle^k\, \xi^{i_1}\dots\xi^{i_m}\, I^k f(x-\langle x,\xi\rangle\xi,\xi)\,\mathrm{dS}_\xi.\end{aligned}$$

Note that for $x\in\mathbb{R}^n$ and $\xi\in\mathbb{S}^{n-1}$, $(x-\langle x,\xi\rangle\xi,\xi)\in T\mathbb{S}^{n-1}$. Since I^k_m and J^k_m agree on $T\mathbb{S}^{n-1}$, we have

$$(N^k_m\, f)_{i_1\dots i_m}(x) = \int\limits_{\mathbb{S}^{n-1}} \langle x,\xi\rangle^k\, \xi_{i_1}\dots\xi_{i_m}\, J^k_m f(x-\langle x,\xi\rangle\xi,\xi)\,\mathrm{dS}_\xi. \tag{2.19}$$

Using (2.14), we have

$$\begin{aligned}(N^k_m\, f)_{i_1\dots i_m}(x) &= \sum_{l=0}^{k}\binom{k}{l}\int\limits_{\mathbb{S}^{n-1}} \langle x,\xi\rangle^{2k-l}\xi_{i_1}\dots\xi_{i_m}(J^l_m f)(x,\xi)\,\mathrm{dS}_\xi\\ &= \sum_{l=0}^{k}\binom{k}{l}\int\limits_{\mathbb{S}^{n-1}}\int\limits_{-\infty}^{\infty} \langle x,\xi\rangle^{2k-l}\,\xi_{i_1}\dots\xi_{i_m}\,t^l\, f_{j_1\dots j_m}(x+t\xi)\xi^{j_1}\dots\xi^{j_m}\,\mathrm{d}t\,\mathrm{dS}_\xi\\ &= 2\sum_{l=0}^{k}\binom{k}{l}\int\limits_{\mathbb{S}^{n-1}}\int\limits_{0}^{\infty} \langle x,\xi\rangle^{2k-l}\,\xi_{i_1}\dots\xi_{i_m}\,t^l\, f_{j_1\dots j_m}(x+t\xi)\xi^{j_1}\dots\xi^{j_m}\,\mathrm{d}t\,\mathrm{dS}_\xi.\end{aligned}$$

Consider the change of variable $x+t\xi=y$ to obtain $t=|y-x|$, $\xi=\dfrac{y-x}{|y-x|}$. We have

$$\begin{aligned}&(N^k_m f)_{i_1\dots i_m}(x)\\ &\quad= 2\sum_{l=0}^{k}\binom{k}{l}\int\limits_{\mathbb{R}^n}\langle x,\frac{y-x}{|y-x|}\rangle^{2k-l}\left(\left(\frac{y-x}{|y-x|}\right)^{\odot 2m}\right)_{i_1\dots i_m j_1\dots j_m} f_{j_1\dots j_m}(y)\,\frac{|y-x|^l}{|y-x|^{n-1}}\,\mathrm{d}y.\end{aligned}$$

Note that for $x,z\in\mathbb{R}^n$, we can write $\langle x,z\rangle^r = j_{x^{\odot r}} z^{\odot r}$. Then we can write

$$(N^k_m f)_{i_1\dots i_m}(x) = 2\sum_{l=0}^{k}\binom{k}{l}\int\limits_{\mathbb{R}^n}\left(j_{x^{\odot 2k-l}}(y-x)^{\odot 2m+2k-l}\right)_{i_1\cdots i_m j_1\cdots j_m}\frac{f_{j_1\dots j_m}(y)}{|y-x|^{2m+2k-2l+n-1}}\,\mathrm{d}y.$$

This gives

$$(N^k_m f)_{i_1\dots i_m}(x) = 2\sum_{l=0}^{k}\binom{k}{l}(-1)^l x^{\odot 2k-l}_{p_1\cdots p_{2k-l}}\left[f_{j_1\dots j_m} * \frac{\left(x^{\odot 2m+2k-l}\right)_{p_1\cdots p_{2k-l} i_1\cdots i_m j_1\dots j_m}}{|x|^{2m+2k-2l+n-1}}\right]. \tag{2.20}$$

Equation (2.20) makes sense for $f\in\mathcal{E}'(\mathbb{R}^n;S^m)$ as well. For $f\in\mathcal{E}'(\mathbb{R}^n;S^m)$, $N^k_m:\mathcal{E}'(\mathbb{R}^n;S^m)\to\mathcal{D}'(\mathbb{R}^n;S^m)$ can be viewed as a multiplication of a smooth function with the convolution of a compactly-supported distribution and a tempered distribution.

We will use in our calculations the divergence of the normal operator of momentum ray transforms given by

$$(\delta N^k_m f)_{i_1\dots i_{m-1}}(x) = k\int\limits_{\mathbb{S}^{n-1}}\langle x,\xi\rangle^{k-1}\,\xi_{i_1}\dots\xi_{i_{m-1}}\,J^k f(x-\langle x,\xi\rangle\xi,\xi)\,\mathrm{dS}_\xi. \tag{2.21}$$

This can be obtained by directly applying the divergence operator to (2.19). Iterating, we get the formula

$$(\delta^r N^k_m f)_{i_1\dots i_{m-r}}(x) = \frac{k!}{(k-r)!}\int\limits_{\mathbb{S}^{n-1}}\langle x,\xi\rangle^{k-r}\,\xi_{i_1}\dots\xi_{i_{m-r}}\,J^k f(x-\langle x,\xi\rangle\xi,\xi)\,\mathrm{dS}_\xi$$

$$= \frac{k!}{(k-r)!} \sum_{l=0}^{k} \binom{k}{l} \int_{\mathbb{S}^{n-1}} \langle x, \xi \rangle^{2k-r-l}\, \xi_{i_1} \dots \xi_{i_{m-r}}\, J^l f(x,\xi)\, \mathrm{dS}_\xi.$$

In particular,

$$(\delta^k N_m^k f)_{i_1 \dots i_{m-k}}(x) = k! \sum_{l=0}^{k} \binom{k}{l} \int_{\mathbb{S}^{n-1}} \langle x, \xi \rangle^{k-l}\, \xi_{i_1} \dots \xi_{i_{m-k}}\, J^l f(x,\xi)\, \mathrm{dS}_\xi, \tag{2.22}$$

and

$$\delta^{k+1} N_m^k f = 0. \tag{2.23}$$

To conclude this section, we remark that our approach for proving a unique continuation principle for the transverse ray transform on symmetric tensor fields is based on the analysis for the ray transform of scalar functions/distributions, and since we already know to handle this case, we do not define transverse ray transform of symmetric tensor field distributions separately.

We are now ready to state the main results of the article.

We say a function ψ *vanishes to infinite order at a point* $x_0 \in \mathbb{R}^n$ if ψ is smooth in a neighborhood of x_0 and ψ along with its partial derivatives of all orders vanishes at x_0, that is, $\partial_x^\alpha \psi(x_0) = 0$ for all multi-indices α.

Theorem 2.1 (UCP for ray transform I)**.** *Let $U \subseteq \mathbb{R}^n$ be any non empty open set and $n \geq 2$. Let $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$ be such that $Rf|_U = 0$ and $N_m f$ vanishes to infinite order at some $x_0 \in U$. Then f is a potential field, that is, there exists a $v \in \mathcal{E}'(\mathbb{R}^n; S^{m-1})$ such that $f = \mathrm{d}v$.*

Theorem 2.2 (UCP for ray transform II)**.** *Let $U \subseteq \mathbb{R}^n$ be any non empty open set and $n \geq 2$. Let $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$ be such that $Rf = 0$ and $N_m f = 0$ in U. Then f is a potential field.*

Corollary 2.3 (UCP for ray transform III)**.** *Let $U \subseteq \mathbb{R}^n$ be open. Let $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$ be such that $Rf|_U = 0$ and the ray transform of f vanishes on all lines passing through U, that is, $J_m f(x,\xi) = 0$ for $x \in U$, $\xi \in \mathbb{S}^{n-1}$. Then f is a potential field.*

Theorem 2.4 (UCP for momentum ray transform I)**.** *Let $U \subseteq \mathbb{R}^n$. Let $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$ be such that for some $0 \leq k \leq m$, $R^k f|_U = 0$. If $N_m^p f|_U = 0$ for all $0 \leq p \leq k$, then f is a generalized potential field, that is, there exists a $v \in \mathcal{E}'(\mathbb{R}^n; S^{m-k-1})$ such that $f = \mathrm{d}^{k+1} v$.*

Theorem 2.5 (UCP for momentum ray transform II)**.** *Let $U \subseteq \mathbb{R}^n$ and $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$. Suppose for some $0 \leq k \leq m$, $R^k f|_U = 0$ and $J_m^k f(x,\xi) = 0$ for all $(x,\xi) \in U \times \mathbb{S}^{n-1}$, then f is a generalized potential tensor field.*

In all the results above the support of v is contained in the convex hull of the support of f.

Theorem 2.6 (UCP for transverse ray transform)**.** *Let $n \geq 3$ and $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$. Assume that $\mathcal{T} f = 0$ along all the lines intersecting a non-empty open set U and $f = 0$ in U. Then $f \equiv 0$.*

3. UCP FOR THE RAY TRANSFORM

In this section, we prove unique continuation properties for the ray transforms. We prove Theorem 2.1. We first show that $N_m f$ is smooth in U if $Rf|_U = 0$.

Lemma 3.1. *Let $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$ be such that $Rf|_U = 0$ for some open set $U \subseteq \mathbb{R}^n$. Then, $N_m f|_U$ is smooth.*

Proof. We use the following formula proved in [3, Theorem 2.5] for $f \in \mathcal{S}(\mathbb{R}^n; S^m)$:

$$\Delta^m ({}^s f) = 2^m \delta_e^m Rf, \tag{3.1}$$

where δ_e is the even indices divergence operator [3, (2.4)] and ${}^s f$ is the solenoidal component of f. In (3.1), the formula is interpreted componentwise. The same proof works for $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$ as well. By hypothesis, the right hand side of (3.1) is 0 in U, and hence $\Delta^m ({}^s f) = 0$. By Weyl's Lemma [21], ${}^s f$ is smooth in U, and hence so is $N_m\, {}^s f$. Since $N_m f = N_m^s f$; see [20], we are done. ☐

Now we will only prove Theorem 2.1. Theorem 2.2 is a trivial consequence. The idea of proof is to reduce the problem to that of the functions case as in [7]. The following proposition serves as the key ingredient.

Proposition 3.2. *For $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$, the following equality holds:*

$$m!\, N_0(Rf)_{i_1 j_1 \dots i_m j_m} = \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} c_{l,m} (R(i^l j^l N_m f))_{i_1 j_1 \dots i_m j_m}, \tag{3.2}$$

where $\lfloor x \rfloor$ denotes the greatest integer function $\leq x$ and the constants $c_{l,m}$ are given by

$$c_{l,m} = \left\{ \prod_{p=0}^{m-l-1} (n-1+2p) \right\} \frac{(-1)^l\, m!}{2^l\, l!\, (m-2l)!}.$$

Remark 3.3. *An inversion formula recovering the Saint-Venant operator of f from its ray transform is proved in* [20, Theorem 2.12.3]. *From* (3.2)*, one can derive this inversion formula. However, our approach here is different and in our opinion simpler than that of* [20, Theorem 2.12.3].

To prove this proposition we first prove a technical result, see Lemma 3.5, which will be used in the next two sections. We first recall the definition of a positive homogeneous function.

A function g is called *positive homogeneous of degree* λ if

$$g(rx) = r^\lambda g(x)$$

for all $x \in \mathbb{R}^n$, and $r > 0$.

The following lemma was recently used in [13] as well.

Lemma 3.4. *If g is smooth and positive homogeneous of degree λ such that $n + \lambda > 0$, then*

$$\int_{\mathbb{S}^{n-1}} g(\xi)\, \mathrm{dS}_\xi = (n+\lambda) \int_{|\xi| \leq 1} g(\xi)\, \mathrm{d}\xi.$$

Proof. Using polar coordinates, we have

$$\begin{aligned} \int_{|\xi|\leq 1} g(\xi)\, \mathrm{d}\xi &= \int_0^1 \int_{\mathbb{S}^{n-1}} g(r\xi) r^{n-1}\, \mathrm{dS}_\xi\, \mathrm{d}r \\ &= \int_0^1 \int_{\mathbb{S}^{n-1}} r^{n+\lambda-1} g(\xi) \mathrm{dS}_\xi\, \mathrm{d}r \\ &= \frac{1}{(n+\lambda)} \int_{\mathbb{S}^{n-1}} g(\xi) \mathrm{dS}_\xi. \end{aligned}$$

□

Lemma 3.5. *Let $n \geq 2$ and g be a smooth function on $\mathbb{R}^n$ such that g is positive homogeneous of degree $s-1$ for some $s \in \mathbb{N}$. Then the following equality holds:*

$$\int_{\mathbb{S}^{n-1}} \frac{\partial^s g}{\partial \xi_{i_1} \dots \partial \xi_{i_s}} \mathrm{dS}_\xi = \sum_{l=0}^{\lfloor \frac{s}{2} \rfloor} c_{l,s} \int_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot s}\right)_{i_1 \cdots i_s} g(\xi)\, \mathrm{dS}_\xi, \tag{3.3}$$

with the constants $c_{l,s}$ given by

$$c_{l,s} = \prod_{w=0}^{(s-l-1)} (n-1+2w) \frac{(-1)^l\, s!}{2^l\, l!\, (s-2l)!}. \tag{3.4}$$

Proof. The proof proceeds by induction on s. We first prove (3.3) for $s = 1$.

For a smooth function g positive homogeneous of degree 0, we have from Lemma 3.4,

$$\int_{\mathbb{S}^{n-1}} \frac{\partial g}{\partial \xi_i}\, \mathrm{dS}_\xi = (n-1) \int_{|\xi|\leq 1} \frac{\partial g}{\partial \xi_i}\, \mathrm{d}\xi,$$

since $\dfrac{\partial g}{\partial \xi_i}$ is positive homogeneous of degree -1. Now applying the divergence theorem, we get,

$$\int\limits_{|\xi|\leq 1} \frac{\partial g}{\partial \xi_i}\,\mathrm{d}\xi = \int\limits_{\mathbb{S}^{n-1}} \xi_i g(\xi)\mathrm{dS}_\xi.$$

Hence

$$\int\limits_{\mathbb{S}^{n-1}} \frac{\partial g}{\partial \xi_i}\,\mathrm{dS}_\xi = (n-1)\int\limits_{\mathbb{S}^{n-1}} \xi_i g(\xi)\mathrm{dS}_\xi. \tag{3.5}$$

For $s=1$, the only choice for l is 0 and $c_{0,1}=n-1$ in (3.4). Thus for $s=1$ (3.3) agrees with (3.5).

Now assume that (3.3) is true for some $s=r-1$. We want to show that the equality (3.3) holds for $s=r$.

Let g be a positive homogeneous of degree $r-1$. Then, $\dfrac{\partial g}{\partial \xi_{i_r}}$ is positive homogeneous of degree $r-2$ and by induction hypothesis we obtain

$$\int\limits_{\mathbb{S}^{n-1}} \frac{\partial^r g}{\partial \xi_{i_1}\dots\partial \xi_{i_r}}\,\mathrm{dS}_\xi = \sum_{l=0}^{\lfloor\frac{r-1}{2}\rfloor} c_{l,r-1}\left(\int\limits_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}} \frac{\partial g}{\partial \xi_{i_r}}\,\mathrm{dS}_\xi\right).$$

Note that $i^l j^l(\xi^{\odot r-1})\dfrac{\partial g}{\partial \xi_{i_r}}$ is a positive homogeneous function of degree $2r-2l-3$. Applying Lemma 3.4 and using Gauss divergence theorem, we get

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}} \frac{\partial^r g}{\partial \xi_{i_1}\dots\partial \xi_{i_r}}\,\mathrm{dS}_\xi = \sum_{l=0}^{\lfloor\frac{r-1}{2}\rfloor} c_{l,r-1}(n+2r-2l-3)\Bigg[&-\int\limits_{|\xi|\leq 1}\left(\frac{\partial}{\partial \xi_{i_r}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}}\right) g(\xi)\,\mathrm{d}\xi \\ &+\int\limits_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}} \xi_{i_r}\, g(\xi)\,\mathrm{dS}_\xi\Bigg].\end{aligned} \tag{3.6}$$

We now use Lemma 3.4 in the first integral in (3.6) and obtain

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}} \frac{\partial^r g}{\partial \xi_{i_1}\dots\partial \xi_{i_r}}\,\mathrm{dS}_\xi &= \sum_{l=0}^{\lfloor\frac{r-1}{2}\rfloor} c_{l,r-1}(n+2r-2l-3)\int\limits_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}} \xi_{i_r} g(\xi)\,\mathrm{dS}_\xi \\ &\quad - \sum_{l=0}^{\lfloor\frac{r-1}{2}\rfloor} c_{l,r-1}\int\limits_{\mathbb{S}^{n-1}}\left(\frac{\partial}{\partial \xi_{i_r}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}}\right) g(\xi)\,\mathrm{dS}_\xi,\end{aligned} \tag{3.7}$$

We separate the $l=0$ term from the first sum, and $l=\lfloor\dfrac{r-1}{2}\rfloor$ term from the second sum in (3.7) to get

$$\begin{aligned}\int\limits_{\mathbb{S}^{n-1}} \frac{\partial^r g}{\partial \xi_{i_1}\dots\partial \xi_{i_r}}\,\mathrm{dS}_\xi &= c_{0,r-1}(n+2r-3)\int\limits_{\mathbb{S}^{n-1}} \xi_{i_1}\dots\xi_{i_r}\, g(\xi)\,\mathrm{dS}_\xi \\ &\quad + \sum_{l=1}^{\lfloor\frac{r-1}{2}\rfloor} c_{l,r-1}(n+2r-2l-3)\int\limits_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}} \xi_{i_r}\, g(\xi)\,\mathrm{dS}_\xi \\ &\quad - \sum_{l=0}^{\lfloor\frac{r-1}{2}\rfloor-1} c_{l,r-1}\int\limits_{\mathbb{S}^{n-1}}\left(\frac{\partial}{\partial \xi_{i_r}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}}\right) g(\xi)\,dS(\xi) \\ &\quad - c_{\lfloor\frac{r-1}{2}\rfloor,r-1}\int\limits_{\mathbb{S}^{n-1}}\left(\frac{\partial}{\partial \xi_{i_r}} i^{\lfloor\frac{r-1}{2}\rfloor} j^{\lfloor\frac{r-1}{2}\rfloor} \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}}\right) g(\xi)\,\mathrm{dS}_\xi.\end{aligned} \tag{3.8}$$

We analyze the integrals in (3.8) separately.

Since $c_{0,r-1}(n+2r-3) = c_{0,r}$, we have

$$c_{0,r-1}(n+2r-3) \int\limits_{\mathbb{S}^{n-1}} \xi_{i_1} \dots \xi_{i_r}\, g(\xi)\, \mathrm{dS}_\xi = c_{0,r} \int\limits_{\mathbb{S}^{n-1}} \xi_{i_1} \dots \xi_{i_r}\, g(\xi)\, \mathrm{dS}_\xi. \tag{3.9}$$

Next we analyze the last term in (3.8). First we consider the case of odd r. Letting $r = 2k+1$, we have $\lfloor \dfrac{r-1}{2} \rfloor = k$. Then we have

$$i^k j^k \left(\xi^{\odot 2k}\right)_{i_1 \cdots i_{2k}} = \sigma(i_1 \cdots i_{2k}) \left(\delta_{i_1 i_2} \cdots \delta_{i_{2k-1} i_{2k}}\right).$$

Hence for r odd, the last term in (3.8) is 0.

Next we consider the case when r is even. Letting $r = 2k$, we have $\lfloor \dfrac{r-1}{2} \rfloor = k-1 = \lfloor \dfrac{r}{2} \rfloor - 1$. This implies

$$\begin{aligned}
c_{\lfloor \frac{r-1}{2} \rfloor, r-1} &= \prod_{w=0}^{r-k-1} (n-1+2w) \frac{(-1)^{k-1}\,(r-1)!}{2^{k-1}\,(k-1)!\,1!} \\
&= -\prod_{w=0}^{r-k-1} (n-1+2w) \frac{(-1)^{k}\, r!}{2^{k}\, k!} \text{ using } \quad r = 2k, \\
&= -c_{k,r} = -c_{\lfloor \frac{r}{2} \rfloor, r}.
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial}{\partial \xi_{i_r}} \left(i^{k-1} j^{k-1} \left(\xi^{\odot 2k-1}\right)_{i_1 \cdots i_{2k-1}} \right) &= \frac{\partial}{\partial \xi_{i_r}} \sigma(i_1 \cdots i_{2k-1}) \left(\delta_{i_1 i_2} \cdots \delta_{i_{2k-3} i_{2k-2}} \xi_{i_{2k-1}}\right) \\
&= \sigma(i_1 \cdots i_{2k-1}) \left(\delta_{i_1 i_2} \cdots \delta_{i_{2k-3} i_{2k-2}} \delta_{i_{2k-1} i_{2k}}\right).
\end{aligned}$$

We observe, recalling that $r = 2k$,

$$\sigma(i_1 \cdots i_{r-1}) \{\delta_{i_1 i_2} \cdots \delta_{i_{r-1} i_r}\} = \sigma(i_1 \cdots i_r) \{\delta_{i_1 i_2} \cdots \delta_{i_{r-1} i_r}\}.$$

Therefore,

$$\frac{\partial}{\partial \xi_{i_r}} \left(i^{k-1} j^{k-1} \left(\xi^{\odot 2k-1}\right)_{i_1 \cdots i_{2k-1}} \right) = i^k j^k \left(\xi^{\odot 2k}\right)_{i_1 \cdots i_{2k}}.$$

Summarizing, we have,

$$\begin{aligned}
&-c_{\lfloor \frac{r-1}{2} \rfloor, r-1} \int\limits_{\mathbb{S}^{n-1}} \frac{\partial}{\partial \xi_{i_r}} \left(i^{\lfloor \frac{r-1}{2} \rfloor} j^{\lfloor \frac{r-1}{2} \rfloor} \left(\xi^{\odot r}\right)_{i_1 \cdots i_r} \right) g(\xi) \mathrm{dS}_\xi \\
&= \begin{cases} 0 & \text{if } r \text{ is odd} \\ c_{\lfloor \frac{r}{2} \rfloor, r} \displaystyle\int\limits_{\mathbb{S}^{n-1}} i^{\lfloor \frac{r}{2} \rfloor} j^{\lfloor \frac{r}{2} \rfloor} \left(\xi^{\odot r}\right)_{i_1 \cdots i_r} g(\xi)\, \mathrm{dS}_\xi & \text{if } r \text{ is even.} \end{cases}
\end{aligned} \tag{3.10}$$

We now consider the remaining terms in (3.7):

$$\begin{aligned}
&\sum_{l=1}^{\lfloor \frac{r-1}{2} \rfloor} c_{l,r-1}(n+2r-2l-3) \int\limits_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1 \cdots i_{r-1}} \xi_{i_r}\, g(\xi)\, \mathrm{dS}_\xi \\
&- \sum_{l=0}^{\lfloor \frac{r-1}{2} \rfloor - 1} c_{l,r-1} \int\limits_{\mathbb{S}^{n-1}} \left(\frac{\partial}{\partial \xi_{i_r}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1 \cdots i_{r-1}} \right) g(\xi)\, \mathrm{dS}_\xi = I \quad \text{(say)}
\end{aligned}$$

Re-indexing the second integral above we obtain

$$\begin{aligned} I = &\sum_{l=1}^{\lfloor\frac{r-1}{2}\rfloor} c_{l,r-1}(n+2r-2l-3)\int_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}} \xi_{i_r}\, g(\xi)\,\mathrm{dS}_\xi \\ &- \sum_{l=1}^{\lfloor\frac{r-1}{2}\rfloor} c_{l-1,r-1}\int_{\mathbb{S}^{n-1}} \left(\frac{\partial}{\partial \xi_{i_r}} i^{l-1}j^{l-1}\left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}}\right) g(\xi)\,\mathrm{dS}_\xi. \end{aligned} \tag{3.11}$$

Note that $c_{l,r-1}(n+2r-2l-3) = \dfrac{r-2l}{r}\, c_{l,r}$ and $-c_{l-1,r-1} = \dfrac{2l}{r(r-2l+1)} c_{l,r}$. Combining this with (3.11) we get

$$\begin{aligned} I = \sum_{l=1}^{\lfloor\frac{r-1}{2}\rfloor} \frac{c_{l,r}}{r}\Bigg\{&(r-2l)\int_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}} \xi_{i_r}\, g(\xi)\,\mathrm{dS}_\xi \\ &+ \frac{2l}{r-2l+1}\int_{\mathbb{S}^{n-1}} \left(\frac{\partial}{\partial \xi_{i_r}}\, i^{l-1}j^{l-1}\left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}}\right) g(\xi)\,\mathrm{dS}_\xi\Bigg\}. \end{aligned}$$

Finally, we observe the following: for $0 \le l \le \lfloor \dfrac{r-1}{2} \rfloor$,

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} i^l j^l \xi^{\odot r}_{i_1\cdots i_r}\, g(\xi)\, d\xi = &\frac{(r-2l)}{r}\int_{\mathbb{S}^{n-1}} i^l j^l \left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}} \xi_{i_r}\, g(\xi)\,\mathrm{dS}_\xi \\ &+ \frac{2l}{r(r-2l+1)}\int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial \xi_{i_r}} \left(i^{l-1}j^{l-1}\left(\xi^{\odot r-1}\right)_{i_1\cdots i_{r-1}}\right) g(\xi)\,\mathrm{dS}_\xi. \end{aligned} \tag{3.12}$$

This can be seen by expanding the left hand side expression,

$$i^l j^l \left(\xi^{\odot r}\right)_{i_1\cdots i_r} = \sigma(i_1\cdots i_r)\left(\delta_{i_1 i_2}\cdots \delta_{i_{2l-1}i_{2l}}\xi_{i_{2l+1}}\cdots \xi_{i_r}\right).$$

The first term comes from those permutations that takes i_r to one of $i_{2l+1}\cdots i_r$ and the second term comes from the complement.

Finally, substituting (3.9), (3.10) and (3.12) into (3.8) we get (3.3) for $s = r$. This completes the proof. $\square$

Proof of Proposition 3.2. We prove (3.2) for $f \in C_c^\infty(\mathbb{R}^n; S^m)$ first.

By an iteration of [20, Equation 2.10.2]

$$(-2)^m m!\, I_0((Rf)_{i_1j_1\cdots i_mj_m}) = J_{i_1j_1}\cdots J_{i_mj_m}(J_m f), \tag{3.13}$$

where the ray transform acts on the scalar function $(Rf)_{i_1j_1\cdots i_mj_m}$ on the left hand side and for each $1 \le i, j \le n$ the John operator [8] J_{ij} is defined, for functions $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n\backslash\{0\})$, by

$$J_{ij}\varphi(x,\xi) = \left(\frac{\partial^2\varphi}{\partial x_i \partial \xi_j} - \frac{\partial^2\varphi}{\partial x_j\partial\xi_i}\right). \tag{3.14}$$

Integrating both sides of (3.13) over $\mathbb{S}^{n-1}$,

$$(-2)^m m! N_0((Rf)_{i_1j_1\dots i_mj_m}) = 2^m \alpha(i_1j_1)\dots\alpha(i_mj_m)\frac{\partial^m}{\partial x_{i_1}\dots\partial x_{i_m}}\int_{\mathbb{S}^{n-1}} \frac{\partial^m J_m f(x,\xi)}{\partial \xi_{j_1}\dots\partial\xi_{j_m}}\,\mathrm{dS}_\xi.$$

Using Lemma 3.5,

$$\begin{aligned} &(-2)^m m! N_0((Rf)_{i_1j_1\dots i_mj_m}) \\ &= 2^m\alpha(i_1j_1)\dots\alpha(i_mj_m)\frac{\partial^m}{\partial x_{i_1}\dots\partial x_{i_m}}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor} c_{l,m}\Big(\int_{\mathbb{S}^{n-1}}\left(i^l j^l\left(\xi^{\odot m}\right)_{j_1\cdots j_m} J_m f(x,\xi)\,\mathrm{dS}_\xi\right)\Big) \end{aligned}$$

$$= 2^m \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} c_{l,m} R\left(i^l j^l N_m f\right)_{i_1 j_1 \cdots i_m j_m}.$$

Now using anti-symmetry of R on the right hand side, we get (3.2). The proof for $f \in C_c^\infty(\mathbb{R}^n; S^m)$ is complete. The proof for $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$ follows by integral representation for the normal operator N_m; see (2.16), the density of $C_c^\infty(\mathbb{R}^n; S^m)$ in $\mathcal{E}'(\mathbb{R}^n; S^m)$ combined with the fact that convolution:

$$(u, v) \to u * v \text{ from } \mathcal{E}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n),$$

is a separately continuous bilinear map. See [22, Theorem 27.6]. $\square$

With the preliminary results in place, we prove the unique continuation results for the ray transform of symmetric tensor fields below.

Proof of Theorem 2.1. By Lemma 3.1, the hypothesis $Rf|_U = 0$ implies that the normal operator $N_m f$ is smooth in U. Also, $N_m f$ vanishes to infinite order at $x_0 \in U$. Thus, by (3.2), $N_0(Rf)_{i_1 j_1 \dots i_m j_m}$ is smooth in U and vanishes to infinite order at $x_0 \in U$. Using [6, Theorem 1.1], we conclude that $(Rf)_{i_1 j_1 \dots j_m j_m} \equiv 0$. Since this is true for all indices $i_1, j_1, \dots i_m, j_m$, Rf vanishes identically on $\mathbb{R}^n$, which is equivalent to Wf vanishing on $\mathbb{R}^n$. Thus by [20, Theorem 2.5.1], there exists a field $v \in \mathcal{E}'(\mathbb{R}^n; S^{m-1})$ whose support is contained in the convex hull of support of f and $f = \mathrm{d}v$. $\square$

The proof of Theorem 2.2 follows as a direct consequence of Theorem 2.1.

Proof of Corollary 2.3. Since $J_m f(x, \xi) = 0$ for $x \in U$ and $\xi \in \mathbb{S}^{n-1}$, using homogeneity properties, we know $I_m f\,(x - \langle x, \xi\rangle \xi, \xi)$ and hence we have that $N_m f(x) = 0$ for all $x \in U$. This in particular implies that $N_m f$ is smooth in U and, trivially, vanishes to infinite order at any point $x_0 \in U$. The conclusion follows from Theorem 2.1. $\square$

We also present a much simpler proof working directly with the ray transform.

Alternate proof of Corollary 2.3. Since $I_m f(x, \xi) = 0$ for all $\xi \in \mathbb{S}^{n-1}$ and $x \in U$, the left hand side of (3.13) vanishes, and hence so does the right hand side. Since $f \in \mathcal{E}'(\mathbb{R}^n; S^m)$, $(Rf)_{i_1 j_1 \dots i_m j_m} \in \mathcal{E}'(\mathbb{R}^n)$ and thus by [6, Theorem 1.2], Rf vanishes identically on $\mathbb{R}^n$. The proof now follows from [20, Theorem 2.5.1]. $\square$

4. UCP FOR MOMENTUM RAY TRANSFORMS

In this section we study unique continuation for momentum ray transforms. We prove an analogue of the identity proved for ray transforms (Proposition 3.2), for momentum ray transforms in Proposition 4.4 below. We first show the equivalence of R^k and W^k and then we prove a lemma required in the proof of Proposition 3.2.

Lemma 4.1. *Let $f \in \mathcal{D}'(\mathbb{R}^n; S^m)$ and $U \subseteq \mathbb{R}^n$ be open. Then for $0 \le k \le m$, $W^k f|_U = 0$ if and only if $R^k f|_U = 0$. In fact, the following equalities hold*

$$(W^k f)_{p_1 \dots p_{m-k} q_1 \dots q_{m-k} i_1 \dots i_k} = 2^{m-k} \sigma(q_1 \dots q_{m-k} i_1 \dots i_k) \sigma(p_1 \dots p_{m-k}) (R^k f)_{p_1 q_1 \dots p_{m-k} q_{m-k} i_1 \dots i_k} \tag{4.1}$$

$$(R^k f)_{p_1 q_1 \dots p_{m-k} q_{m-k} i_1 \dots i_k} = \frac{1}{m-k+1} \binom{m}{k} \alpha(p_1 q_1) \dots \alpha(p_{m-k} q_{m-k}) (W^k f)_{p_1 \dots p_{m-k} q_1 \dots q_{m-k} i_1 \dots i_k}. \tag{4.2}$$

Proof. We present the proof using similar ideas from [20, Lemma 2.4.2]. From [16, Equation 27] we obtain

$$(W^k f)_{p_1 \dots p_{m-k} q_1 \dots q_{m-k} i_1 \dots i_k} = \sigma(q_1 \dots q_{m-k} i_1 \dots i_k) (W f^{i_1 \dots i_k})_{p_1 \dots p_{m-k} q_1 \dots q_{m-k}}.$$

Using (2.6) for an $m - k$-tensor field $f^{i_1 \dots i_k}$ we get (4.1).

Now, we prove (4.2). Decomposing the symmetrizations $\sigma(p_1\dots p_{m-k})$ and $\sigma(q_1\dots q_{m-k}i_1\dots i_k)$ in the definition of W^k with respect to the indices p_{m-k} and q_{m-k} and taking into account the symmetries of f and mixed partial derivatives, we get

$$(W^kf)_{p_1\dots p_{m-k}q_1\dots q_{m-k}i_1\dots i_k}=\sigma(p_1\dots p_{m-k-1})\sigma(q_1\dots q_{m-k-1}i_1\dots i_k)\sum_{l=0}^{m-k}(-1)^l\binom{m-k}{l}\frac{1}{m(m-k)}$$
$$\times\left[(m-k-l)^2\frac{\partial^{m-k}f^{i_1\dots i_k}_{p_1\dots p_{m-k-l-1}p_{m-k}q_1\dots q_l}}{\partial x_{p_{m-k-l}}\dots\partial x_{p_{m-k-1}}\partial x_{q_{l+1}}\dots\partial x_{q_{m-k}}}+l^2\frac{\partial^{m-k}f^{i_1\dots i_k}_{p_1\dots p_{m-k-l}q_1\dots q_{l-1}q_{m-k}}}{\partial x_{p_{m-k-l+1}}\dots\partial x_{p_{m-k}}\partial x_{q_l}\dots\partial x_{q_{m-k-1}}}+h\right],$$

where h is some tensor symmetric in p_{m-k},q_{m-k}. Applying the operator $\alpha(p_{m-k}\,q_{m-k})$ to this equality and noting that it commutes with $\sigma(p_1\dots p_{m-k-1})$ and $\sigma(q_1\dots q_{m-k-1}i_1\dots i_k)$, we obtain

$$\alpha(p_{m-k}\,q_{m-k})(W^kf)_{p_1\dots p_{m-k}q_1\dots q_{m-k}i_1\dots i_k}=\alpha(p_{m-k}\,q_{m-k})\sigma(p_1\dots p_{m-k-1})\sigma(q_1\dots q_{m-k-1}i_1\dots i_k)$$
$$\sum_{l=0}^{m-k}(-1)^l\binom{m-k}{l}\frac{1}{m(m-k)}$$
$$\times\left[(m-k-l)^2\frac{\partial^{m-k}f^{i_1\dots i_k}_{p_1\dots p_{m-k-l-1}p_{m-k}q_1\dots q_l}}{\partial x_{p_{m-k-l}}\dots\partial x_{p_{m-k-1}}\partial x_{q_{l+1}}\dots\partial x_{q_{m-k}}}-l^2\frac{\partial^{m-k}f^{i_1\dots i_k}_{p_1\dots p_{m-k-l}p_{m-k}q_1\dots q_{l-1}}}{\partial x_{p_{m-k-l+1}}\dots\partial x_{p_{m-k-1}}\partial x_{q_l}\dots\partial x_{q_{m-k}}}\right],$$

where we interchanged p_{m-k} and q_{m-k} in the last term, contributing to the negative sign.

Combining the first summand in the brackets of the l-th term of the sum with the second summand of the $(l+1)$-th term, we get

$$\begin{aligned}&\alpha(p_{m-k}\,q_{m-k})(W^kf)_{p_1\dots p_{m-k}q_1\dots q_{m-k}i_1\dots i_k}\\=&\frac{m-k+1}{m}\alpha(p_{m-k}\,q_{m-k})\sigma(p_1\dots p_{m-k-1})\sigma(q_1\dots q_{m-k-1}i_1\dots i_k)\\&\sum_{l=0}^{m-k-1}(-1)^l\binom{m-k-1}{l}\frac{\partial^{m-k}f^{i_1\dots i_k}_{p_1\dots p_{m-k-l-1}p_{m-k}}q_1\dots q_l}{\partial x_{p_{m-k-l}}\dots\partial x_{p_{m-k-1}}\partial x_{q_{l+1}}\dots\partial x_{q_{m-k}}}\\=&\frac{m-k+1}{m}\alpha(p_{m-k}q_{m-k})\frac{\partial}{\partial x_{q_{m-k}}}(W^kf^{p_{m-k}})_{p_1\dots p_{m-k-1}q_1\dots q_{m-k-1}i_1\dots i_k},\end{aligned}$$

where on the right hand side W^k acts on the $(m-1)$-tensor field obtained by fixing the index p_{m-k} in f. Iterating this $(m-k)$-times, we get

$$\alpha(p_1q_1)\dots\alpha(p_{m-k}q_{m-k})(W^kf)_{p_1\dots p_{m-k}q_1\dots q_{m-k}i_1\dots i_k}=\frac{(m-k+1)}{\binom{m}{k}}\alpha(p_1q_1)\dots\alpha(p_{m-k}q_{m-k})\frac{\partial^{m-k}}{\partial x_{q_1}\dots\partial x_{q_{m-k}}}(W^kf^{p_1\dots p_{m-k}})_{i_1\dots i_k}. \tag{4.3}$$

Finally, note that on the right hand side, W^k acts on the k-tensor obtained by fixing $(m-k)$-indices. Since by our convention, W^k acting on k-tensors is just the identity operator, (4.2) follows. □

Remark 4.2. *From the expression for R^k in (2.9), it is clear that given R^k for some $0\le k\le m$, we can recover R^s for all $0\le s<k$. Since W^k and R^k are equivalent, we can recover W^s for $0\le s\le k$ as well.*

Lemma 4.3. *Let $f\in C_c^\infty(\mathbb{R}^n;S^m)$. For $m\ge 0$ and $0\le k\le m$,*

$$\int\limits_{\mathbb{S}^{n-1}}\xi_{i_1}\dots\xi_{i_{m-k}}\,(J^k_mf)(x,\xi)\,\mathrm{dS}_\xi=\sum_{r=0}^{k}(-1)^{k-r}\frac{1}{r!}\binom{k}{r}\,(j_{x^{\odot k-r}}\delta^rN^r_mf)_{i_1\cdots i_{m-k}}\,, \tag{4.4}$$

where the operator j is given in (2.2) and δ is given in (2.3).

Proof. We prove this result by induction on k. For $k=0$, (4.4) is just the definition of the normal operator of the ray transform of a symmetric m-tensor field f given in (2.19).

Now assume that (4.4) is true for all $0,1,\dots,k-1$ and we want to prove this for k. By (2.22), we have,

$$\int\limits_{\mathbb{S}^{n-1}} \xi_{i_1}\dots\xi_{i_{m-k}}\,(J^k_m f)(x,\xi)\,\mathrm{dS}_\xi = \frac{1}{k!}(\delta^k N^k_m f)_{i_1\dots i_{m-k}}(x) - \sum_{l=0}^{k-1}\binom{k}{l}\int\limits_{\mathbb{S}^{n-1}}\langle x,\xi\rangle^{k-l}\xi_{i_1}\dots\xi_{i_{m-k}}(J^l_m f)(x,\xi)\,\mathrm{dS}_\xi.$$

Since $\langle x,\xi\rangle^r = j_{x^{\odot r}}\xi^{\odot r}$, we have, together with the induction hypothesis,

$$\begin{aligned}
&\int\limits_{\mathbb{S}^{n-1}} \xi_{i_1}\dots\xi_{i_{m-k}}\,(J^k_m f)(x,\xi)\,\mathrm{dS}_\xi\\
&\quad= \frac{1}{k!}(\delta^k N^k_m f)_{i_1\dots i_{m-k}}(x) - \sum_{l=0}^{k-1}\binom{k}{l} j_{x^{\odot k-l}}\left(\int\limits_{\mathbb{S}^{n-1}}\xi_{i_1}\dots\xi_{i_{m-k}}\xi^{\odot k-l}(J^l_m f)(x,\xi)\mathrm{dS}_\xi\right)\\
&\quad= \frac{1}{k!}(\delta^k N^k_m f)_{i_1\dots i_{m-k}}(x) - \sum_{l=0}^{k-1}\binom{k}{l}\sum_{r=0}^{l}(-1)^{l-r}\frac{1}{r!}\binom{l}{r}\left(j_{x^{\odot k-r}}\delta^r N^r_m f\right)_{i_1\dots i_{m-k}}
\end{aligned}$$

Interchanging the order of summation in the second term,

$$= \frac{1}{k!}(\delta^k N^k_m f)_{i_1\dots i_{m-k}}(x) - \sum_{r=0}^{k-1}\frac{1}{r!}\left(j_{x^{\odot k-r}}\delta^r N^r_m f\right)_{i_1\cdots i_{m-k}}\sum_{l=r}^{k-1}\binom{k}{l}(-1)^{l-r}\binom{l}{r}. \tag{4.5}$$

We note that

$$\sum_{l=r}^{k}(-1)^{l-r}\binom{k}{l}\binom{l}{r} = 0. \tag{4.6}$$

This is obtained by differentiating the binomial expansion of $(1+x)^k$, r number of times and letting $x=-1$. From (4.6), we have $\sum_{l=r}^{k-1}\binom{k}{l}(-1)^{l-r}\binom{l}{r} = (-1)^{k-r+1}\binom{k}{r}$. Combining this with (4.5) we get

$$\int\limits_{\mathbb{S}^{n-1}} \xi_{i_1}\dots\xi_{i_{m-k}}\,(J^k_m f)(x,\xi)\,\mathrm{dS}_\xi = \sum_{r=0}^{k}(-1)^{k-r}\frac{1}{r!}\binom{k}{r}\left(j_{x^{\odot k-r}}\delta^r N^r_m f\right)_{i_1\cdots i_{m-k}}.$$

This completes the proof. ☐

Proposition 4.4. *For $f\in\mathcal{E}'(\mathbb{R}^n;S^m)$,*

$$\begin{aligned}
&m!\,N_0((Rf^{i_1\cdots i_k})_{p_1q_1\dots p_{m-k}q_{m-k}})\\
&\quad= \sigma(i_1\dots i_k)\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{\partial^r}{\partial x_{i_1}\dots\partial x_{i_r}}(R^k(G_{m-r}))_{p_1q_1\dots p_{m-k}q_{m-k}i_{r+1}\dots i_k},
\end{aligned}\tag{4.7}$$

where G_{m-r} is a symmetric $(m-r)$-tensor given as

$$(G_{m-r})_{i_{r+1}\dots i_k q_1\dots q_{m-k}} = \sum_{l=0}^{\lfloor\frac{m-r}{2}\rfloor} c_{l,m-r}\, i^l j^l\left(\sum_{p=0}^{r}(-1)^{r-p}\frac{1}{p!}\binom{r}{p}(j_{x^{\odot r-p}}\delta^p N^p_m f)_{i_{r+1}\dots i_k q_1\dots q_{m-k}}\right).$$

Proof. As in the proof of Proposition 3.2, we prove the identity for $f\in C^\infty_c(\mathbb{R}^n;S^m)$.

We begin with the following relation which can be shown in exactly the same way as in (3.13) by considering the ray transform of the $m-k$ symmetric tensor field obtained by fixing the indices $i_1,\cdots,i_k$ and then applying the John operator $m-k$ times

$$(m-k)!(-2)^{m-k}I_0((Rf^{i_1\dots i_k})_{p_1q_1\dots p_{m-k}q_{m-k}}) = J_{p_1q_1}\cdots J_{p_{m-k}q_{m-k}}I_{m-k}\left(f^{i_1\dots i_k}\right). \tag{4.8}$$

Substituting the expression for $J^0_{m-k} f^{i_1\dots i_k}$ from [16, Lemma 4.2] we get

$$\begin{aligned}(-2)^{m-k}(m-k)!\, J^0_0((Rf^{i_1\dots i_k})_{p_1q_1\dots p_{m-k}q_{m-k}})\\ = 2^{m-k}\alpha(p_1q_1)\dots\alpha(p_{m-k}q_{m-k})\frac{\partial^{2m-2k}}{\partial x_{p_1}\dots\partial x_{p_{m-k}}\partial\xi_{q_1}\dots\partial\xi_{q_{m-k}}}\\ \left[\frac{(m-k)!}{m!}\sigma(i_1\dots i_k)\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{\partial^k J^r_m f}{\partial x_{i_1}\dots\partial x_{i_r}\partial\xi_{i_{r+1}}\dots\partial\xi_{i_k}}\right].\end{aligned}$$

Integrating both sides over $\mathbb{S}^{n-1}$, we get

$$\begin{aligned}(-1)^{m-k}(m-k)!\, N_0((Rf^{i_1\dots i_k})_{p_1q_1\dots p_{m-k}q_{m-k}})\\ = \alpha(p_1q_1)\dots\alpha(p_{m-k}q_{m-k})\frac{(m-k)!}{m!}\sigma(i_1\dots i_k)\\ \left[\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{\partial^{m-k+r}}{\partial x_{i_1}\dots\partial x_{i_r}\partial x_{p_1}\dots\partial x_{p_{m-k}}}\right.\\ \left.\left\{\int_{\mathbb{S}^{n-1}}\frac{\partial^{m-r}J^r_m f}{\partial\xi_{i_{r+1}}\dots\partial\xi_{i_k}\partial\xi_{q_1}\dots\partial\xi_{q_{m-k}}}\,\mathrm{dS}_\xi\right\}\right].\end{aligned}$$

Since $J^r_m f$ is a homogeneous function of degree $m-r-1$ in the ξ variable, using Lemma 3.5, we have

$$\begin{aligned}(-1)^{m-k}(m-k)!\, N^0_0((Rf^{i_1\dots i_k})_{p_1q_1\dots p_{m-k}q_{m-k}})\\ = \alpha(p_1q_1)\dots\alpha(p_{m-k}q_{m-k})\frac{(m-k)!}{m!}\sigma(i_1\dots i_k)\times\\ \left[\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{\partial^{m-k+r}}{\partial x_{i_1}\dots\partial x_{i_r}\partial x_{p_1}\dots\partial x_{p_{m-k}}}\right.\\ \left.\left\{\sum_{l=0}^{\lfloor\frac{m-r}{2}\rfloor}c_{l,m-r}\, i^l j^l\int_{\mathbb{S}^{n-1}}\xi_{i_{r+1}}\dots\xi_{i_k}\xi_{q_1}\dots\xi_{q_{m-k}}J^r_m f\,\mathrm{dS}_\xi\right\}\right],\end{aligned}$$

where the constants $c_{l,m-r}$ are given in Lemma 3.5.

Finally, we use Lemma 4.3 to express the last integral in terms of the normal operator,

$$\begin{aligned}(m-k)!\,(-1)^{m-k}N^0_0((Rf^{i_1\dots i_k})_{p_1q_1\dots p_{m-k}q_{m-k}})\\ = \alpha(p_1q_1)\dots\alpha(p_{m-k}q_{m-k})\frac{(m-k)!}{m!}\sigma(i_1\dots i_k)\times\\ \left[\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{\partial^{m-k+r}}{\partial x_{i_1}\dots\partial x_{i_r}\partial x_{p_1}\dots\partial x_{p_{m-k}}}\right.\\ \left.\left\{\sum_{l=0}^{\lfloor\frac{m-r}{2}\rfloor}c_{l,m-r}\, i^l j^l\Big(\sum_{p=0}^{r}(-1)^{r-p}\frac{1}{p!}\binom{r}{p}(j_{x^{\odot r-p}}\delta^p N^p_m f)_{i_{r+1}\dots i_k q_1\dots q_{m-k}}\Big)\right\}\right]\\ = \alpha(p_1q_1)\dots\alpha(p_{m-k}q_{m-k})\frac{(m-k)!}{m!}\sigma(i_1\dots i_k)\times\\ \left[\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{\partial^{m-k+r}}{\partial x_{i_1}\dots\partial x_{i_r}\partial x_{p_1}\dots\partial x_{p_{m-k}}}(G_{m-r})_{q_1\dots q_{m-k}i_{r+1}\dots i_k}\right].\end{aligned}$$

Since $\alpha(p_jq_j)$ for $1\le j\le m-k$ commutes with $\sigma(i_1\dots i_k)$, we take $\alpha(p_jq_j)$ inside the summation, and use the anti-symmetry of R^k to finish the proof. $\square$

With these preliminaries, let us prove Theorems 2.4 and 2.5.

Proof of Theorem 2.4. Since $N_m^p f|_U = 0$ for $0 \le p \le k$, the right hand side of (4.7) vanishes in U. This implies

$$N_0((Rf^{i_1\cdots i_k})_{p_1q_1\cdots p_{m-k}q_{m-k}}) = 0 \quad \text{in} \quad U.$$

By the definition in (2.9), $R^k f|_U = 0$ implies

$$(Rf^{i_1\cdots i_k})_{p_1q_1\cdots p_{m-k}q_{m-k}} = 0 \quad \text{in} \quad U.$$

Applying unique continuation for the normal operator of the ray transform of scalar functions [6, Theorem 1.1] on $(Rf^{i_1\cdots i_k})_{p_1q_1\cdots p_{m-k}q_{m-k}}$, we conclude that

$$(Rf^{i_1\cdots i_k})_{p_1q_1\cdots p_{m-k}q_{m-k}} = 0 \quad \text{in} \quad \mathbb{R}^n,$$

for all $1 \le i_1, \ldots, i_k, p_1, \ldots, p_{m-k}, q_1, \ldots, q_{m-k} \le n$. Again using (2.9), we get $R^k f = 0$ in $\mathbb{R}^n$. Combining Lemma 4.1 and [20, Theorem 2.17.2], we conclude the proof. $\square$

Proof of Theorem 2.5. By [2, Lemma 4.8], we have that $J_m^k f$ determines $J_m^r f$ for all $r < k$. Hence $I_m^0 f, \cdots, I_m^k f$ are determined on $\left(U \times \mathbb{S}^{n-1}\right) \cap T\mathbb{S}^{n-1}$. This then implies that we know $N_m^p f|_U$ for all $0 \le p \le k$. Now using Theorem 2.4, we have the result. $\square$

5. UCP FOR TRANSVERSE RAY TRANSFORM

Proof of Theorem 2.6. We proceed by an argument similar to the one used in [20]. Let f be a compactly supported tensor field distribution. Fix a non-zero vector $\eta \perp \xi$, and consider the compactly supported distribution:

$$\varphi_\eta(x) = f_{i_1\cdots i_m}(x)\eta_{i_1}\cdots\eta_{i_m}.$$

The ray transform of φ_η is well-defined. Fix $x \in \operatorname{supp} f$. Denote $V_H = V \cap H_\eta$, where H_η is a hyperplane with normal η and passing through x. Note that V_H is an open set in $\mathbb{R}^{n-1}$. For $x \in V_H$, define $\varphi_\eta(x) = f_{i_1\cdots i_m}(x)\eta_{i_1}\cdots\eta_{i_m}$ for a fixed η. From the knowledge of the transverse ray transform $\mathcal{T}f$, we have that

$$I^0(\phi_\eta) = 0 \quad \text{and} \quad \phi_\eta = 0 \quad \text{in} \quad V_H.$$

Using unique continuation for scalar functions by [6] we get $\phi_\eta = 0$ in H_η. We can vary η in an open cone $\mathcal{C}$ (say) and obtain $\phi_\eta = 0$ for all $\eta \in \mathcal{C}$. Any such cone always contains n linearly independent vectors say $\eta_1, \cdots, \eta_n$. Then the collection of $\binom{m+n-1}{m}$ symmetric tensors

$$A = \{\eta_{i_1} \odot \eta_{i_2} \odot \cdots \odot \eta_{i_m} : 1 \le i_1, \cdots i_m \le n\}$$

are linearly independent. This can be proved directly; see also [12, Lemma 5.4]. This gives

$$\langle f, \eta^{\odot m}\rangle = 0 \quad \text{for all} \quad \eta \in A,$$

which in turn gives $f(x) = 0$ for fixed x. Varying $x \in \operatorname{supp} f$ we get $f \equiv 0$. $\square$

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