

## Microlocal Analysis of an Ultrasound Transform with Circular Source and Receiver Trajectories

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ABSTRACT. We consider a generalized Radon transform that is used in ultrasound reflection tomography. In our model, the ultrasound emitter and receiver move at a constant distance apart along a circle. We analyze the microlocal properties of the transform  $\mathcal{R}$  that arises from this model. As a consequence, we show that, for distributions with support sufficiently inside the circle,  $\mathcal{R}^*\mathcal{R}$  is an elliptic pseudodifferential operator. We provide a local filtered back projection algorithm,  $\mathcal{L} = \mathcal{R}^*D\mathcal{R}$  where  $D$  is a well-chosen differential operator. We prove that  $\mathcal{L}$  is an elliptic pseudodifferential operator of order one and so for  $f \in \mathcal{E}'(D_b)$ ,  $\mathcal{L}f$  shows all singularities of  $f$ , and we provide reconstructions illustrating this point.

### 1. Introduction

Ultrasound reflection tomography (URT) is one of the safest and most cost effective modern medical imaging modalities (e.g., see [13, 14, 15, 16] and the references there). During its scanning process, acoustic waves emitted from a source reflect from inhomogeneities inside the body, and their echoes are measured by a receiver. This measured data is then used to recover the unknown ultrasonic reflectivity function, which is used to generate cross-sectional images of the body.

In a typical setup of ultrasound tomography, the emitter and receiver are combined into one device (transducer). The transducer emits a short acoustic pulse into the medium, and then switches to receiving mode, recording echoes as a function of time. Assuming that the medium is weakly reflecting (i.e., neglecting multiple

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reflections), and that the speed of sound propagation  $c$  is constant<sup>1</sup>, the echoes measured at time  $t$  uniquely determine the integrals of the reflectivity function over concentric spheres centered at the transducer location and radii  $r = ct/2$  (see Fig. 1 (a) below, [16] and the references there). By focusing the transducer one can consider echoes coming only from a certain plane, hence measuring the integrals of the reflectivity function in that plane along circles centered at the transducer location [15]. Moving the transducer along a curve on the edge of the body, and repeating the measurements one obtains a two-dimensional family of integrals of the unknown function along circles. Hence the problem of image reconstruction in URT can be mathematically reduced to the problem of inverting a circular Radon transform, which integrates an unknown function of two variables along a two-dimensional family of circles.

In the case when the emitter and receiver are separated, the echoes recorded by a transducer correspond to the integrals of the reflectivity function along confocal ellipses. The foci of these ellipses correspond to the locations of the emitter and receiver moving along a fixed curve. While this more general setup has been gaining popularity in recent years (e.g., see [13, 14]), the mathematical theory related to elliptical Radon transforms is relatively undeveloped.

In this paper we consider a setup where the separated emitter and receiver move along a circle at a fixed distance apart (see Fig. 1 (b)). The circular trajectory of their motion is both the simplest case mathematically and the one most often used in practice. By using a dilation and translation, we can assume the circle has radius  $r = 1$  centered at 0. We study the microlocal properties of transform  $\mathcal{R}$  which integrates an unknown function along this family of ellipses.

We prove that  $\mathcal{R}$  is an elliptic Fourier integral operator (FIO) of order  $-1/2$  using the microlocal framework of Guillemin and Guillemin-Sternberg [6, 8] for generalized Radon transforms. We use this to understand when the imaging operator  $\mathcal{R}^*\mathcal{R}$  is a pseudodifferential operator. Specifically, we show that for distributions supported in a smaller disc (the disc  $D_b$  of (2.2)), a microlocal condition introduced by Guillemin [6], the so called Bolker assumption, is satisfied and, consequently, for such distributions  $\mathcal{R}^*\mathcal{R}$  is an elliptic pseudodifferential operator. We construct a differential operator  $D$  such that  $\mathcal{R}^*D\mathcal{R}$  is elliptic of order one. From the tomographic point of view this means that using the measured data one can stably recover all singularities of objects supported inside that disc. We provide reconstructions that illustrate this. We remark that  $\mathcal{R}$  does not recover all singularities in the complement of  $D_b$  and in this sense  $D_b$  is optimal. Stefanov and Uhlmann [18] show for a related problem in monostatic radar that singularities on one side can cancel singularities on the other.

In Section 2, we introduce the basic notation and microlocal analysis as well as Guillemin's framework for understanding Radon transforms. In Section 3, we present the microlocal regularity theorem, and in Section 4 we present reconstructions from a local filtered backprojection algorithm (see equation (4.1)) that illustrates the conclusion of the main theorem. The proof of the microlocal regularity theorem is in Section 5.

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<sup>1</sup>This assumption is reasonable in ultrasound mammography, since the speed of sound is almost constant in soft tissue.

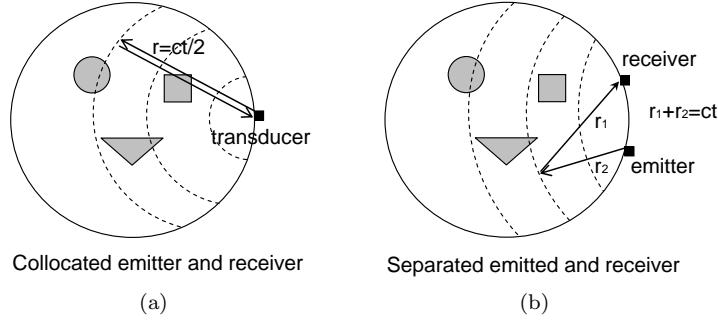


FIGURE 1. A sketch of integrating curves in URT

## 2. Definitions and Preliminaries

We will first define the elliptical Radon transform we consider, provide the general framework for the microlocal analysis of this transform, and show that our transform fits within this framework.

**2.1. The Elliptical Transform.** Recall that in the URT model we consider in this paper, the emitter and receiver move along the circle of radius 1 centered at 0 and are at a fixed distance apart. We denote the fixed difference between the polar angles of emitter and receiver by  $2\alpha$ , where  $\alpha \in (0, \pi/2)$  (see Fig. 2) and define

$$(2.1) \quad a = \sin \alpha, \quad b = \cos \alpha.$$

As we will see later our main result relies on the assumption that the support of the function is small enough. More precisely, we will assume our function is supported in the ball

$$(2.2) \quad D_b = \{x \in \mathbb{R}^2 \mid |x| < b\}.$$

We parameterize the trajectories of the transmitter (emitter) and receiver, respectively, as

$$\begin{aligned} \gamma_T(s) &= (\cos(s - \alpha), \sin(s - \alpha)) \\ \gamma_R(s) &= (\cos(s + \alpha), \sin(s + \alpha)) \quad \text{for } s \in [0, 2\pi]. \end{aligned}$$

Thus, the emitter and receiver rotate around the unit circle and are always  $2a$  units apart. For  $s \in [0, 2\pi]$  and  $L > 2a$ , let

$$E(s, L) = \{x \in \mathbb{R}^2 \mid |x - \gamma_T(s)| + |x - \gamma_R(s)| = L\}.$$

Note that the center of the ellipse  $E(s, L)$  is  $(b \cos s, b \sin s)$  and  $L$  is the diameter of the major axis of  $E(s, L)$ , the so called *major diameter*. This is why we require  $L$  to be greater than the distance between the foci,  $2a$ . As a function of  $s$ , the ellipse  $E(s, L)$  is  $2\pi$ -periodic, and so we will identify  $s \in [0, 2\pi]$  with the point  $(\cos s, \sin s)$  on the unit circle when convenient.

Let

$$Y = \{(s, L) \mid s \in [0, 2\pi], L > 2a\},$$

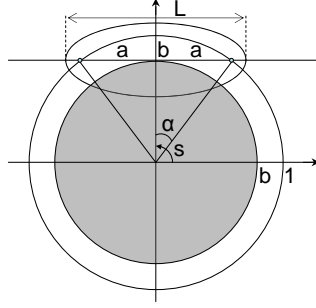


FIGURE 2. A sketch of the domain and the notations

then  $Y$  is the set of parameters for the ellipses.

Let  $(s, L) \in Y$ . The elliptical Radon transform of a locally integrable function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined as

$$\mathcal{R}f(s, L) = \int_{x \in E(s, L)} f(x) dt(x)$$

where  $dt$  is the arc length measure on the ellipse  $E(s, L)$ . The backprojection transform is defined for  $g \in C_c(Y)$  as

$$\mathcal{R}^*g(x) = \int_{s \in [0, 2\pi]} g(s, |x - \gamma_R(s)| + |x - \gamma_T(s)|) w(s, x) ds$$

where the positive smooth weight  $w(s, x)$  is chosen so that  $\mathcal{R}^*$  is the  $L^2$  adjoint of  $\mathcal{R}$  with measure  $dx$  on  $D_b$  and  $ds dL$  on  $Y$  (see equation (2.12) in Example 2.2).

We can compose  $\mathcal{R}$  and  $\mathcal{R}^*$  for the following reasons. If  $f \in \mathcal{D}(D_b)$  then  $\mathcal{R}f$  has compact support in  $Y$  since  $\mathcal{R}f(s, L)$  is zero for  $L$  near  $2a$ . Clearly,  $\mathcal{R} : \mathcal{D}(D_b) \rightarrow \mathcal{D}(Y)$  is continuous so  $\mathcal{R}^* : \mathcal{D}'(Y) \rightarrow \mathcal{D}'(D_b)$  is continuous. Since,  $\mathcal{R}^*$  integrates over a compact set  $[0, 2\pi]$ ,  $\mathcal{R}^* : \mathcal{E}(Y) \rightarrow \mathcal{E}(D_b)$  is continuous, so  $\mathcal{R} : \mathcal{E}'(D_b) \rightarrow \mathcal{E}'(Y)$  is continuous. Therefore,  $\mathcal{R}^*$  can be composed with  $\mathcal{R}$  on domain  $\mathcal{E}'(D_b)$ . Using the parametrization of ellipses  $(s, L)$  one sees that  $\mathcal{R}^*g(x)$  integrates (with a smooth measure) over the set of all ellipses passing through  $x$ .

**2.2. Microlocal Definitions.** We now introduce some notation so we can describe our operators microlocally. Let  $X$  and  $Y$  be smooth manifolds and let

$$\mathcal{C} \subset T^*(Y) \times T^*(X),$$

then we let

$$\mathcal{C}' = \{(y, \eta, x, \xi) \mid (y, \eta, x, -\xi) \in \mathcal{C}\}.$$

The transpose relation is  $\mathcal{C}^t \subset T^*(X) \times T^*(Y)$ :

$$\mathcal{C}^t = \{(x, \xi, y, \eta) \mid (y, \eta, x, \xi) \in \mathcal{C}\}$$

If  $\mathcal{D} \subset T^*(X) \times T^*(Y)$ , then the composition  $\mathcal{D} \circ \mathcal{C}$  is defined

$$\begin{aligned} \mathcal{D} \circ \mathcal{C} &= \{(x', \xi', x, \xi) \mid \exists (y, \eta) \in T^*(Y) \\ &\quad \text{with } (x', \xi', y, \eta) \in \mathcal{D}, (y, \eta, x, \xi) \in \mathcal{C}\}. \end{aligned}$$

**2.3. The Radon Transform and Double Fibrations in General.** Guillemin first put the Radon transform into a microlocal framework, and we now describe this approach and explain how our transform  $\mathcal{R}$  fits into this framework. We will use this approach to prove Theorem 3.1.

Guillemin used the ideas of pushforwards and pullbacks to define Radon transforms and show they are Fourier integral operators (FIOs) in the technical report [5], and these ideas were outlined in [8, pp. 336-337, 364-365] and summarized in [6]. He used these ideas to define FIOs in general in [7, 8]. The dependence on the measures and details of the proofs for the case of equal dimensions were given in [17].

Given smooth connected manifolds  $X$  and  $Y$  of the same dimension, let  $Z \subset Y \times X$  be a smooth connected submanifold of codimension  $k < \dim(X)$ . We assume that the natural projections

$$(2.3) \quad \begin{array}{ccc} & Z & \\ \pi_L \swarrow & & \searrow \pi_R \\ Y & & X \end{array}$$

are both fiber maps. In this case, we call (2.3) a *double fibration*. This framework was used by Helgason [9] to define Radon transforms in a group setting, and it was later generalized to manifolds without a group structure [4].

Following Guillemin and Sternberg, we assume that  $\pi_R$  is a *proper map*; that is, the fibers of  $\pi_R : Z \rightarrow X$  are compact.

The double fibration allows us to define sets of integration for the Radon transform and its dual as follows. For each  $y \in Y$  let

$$E(y) = \pi_R(\pi_L^{-1}(\{y\})),$$

then  $E(y)$  is a subset of  $X$  that is diffeomorphic to the fiber of  $\pi_L : Z \rightarrow Y$ . For each  $x \in X$  let

$$F(x) = \pi_L(\pi_R^{-1}(\{x\})),$$

then  $F(x) \subset Y$  is diffeomorphic to the fiber of  $\pi_R : Z \rightarrow X$ . Since  $\pi_R$  is proper,  $F(x)$  is compact.

Guillemin defined the Radon transform and its dual using pushforwards and pullbacks. The pullback of the function  $f \in C^\infty(X)$  is

$$\pi_R^* f(z) = f(\pi_R(z))$$

and the pushforward to  $X$  of a measure  $\nu$  on  $Z$  is the measure satisfying

$$\int_X f(x) d\pi_{R*}(\nu) = \int_Z (\pi_R^* f) d\nu.$$

The pushforward and pullback for  $\pi_L$  are defined similarly.

By choosing smooth nowhere zero measures  $\mu$  on  $Z$ ,  $m$  on  $X$ , and  $n$  on  $Y$ , one defines the generalized Radon transform of  $f \in C_c^\infty(X)$  as the function  $Rf$  for which

$$(2.4) \quad (Rf)n = \pi_{L*}((\pi_R^* f)\mu).$$

The dual transform for  $g \in C^\infty(Y)$  is the function  $R^*g$  for which

$$(R^*g)m = \pi_{R*}((\pi_L^* g)\mu).$$

This definition is natural because  $R^*$  is automatically the dual to  $R$  by the duality between pushforwards and pullbacks.

The measures  $\mu$ ,  $n$  and  $m$  give the measures of integration for  $R$  and  $R^*$  as follows. Since  $\pi_L : Z \rightarrow Y$  is a fiber map locally above  $y \in Y$ , the measure  $\mu$  can be written as a product of the measure  $n$  and a smooth measure on the fiber. This fiber is diffeomorphic to  $E(y)$ , and the measure on the fiber can be pushed forward using this diffeomorphism to a measure  $\mu_y$  on  $E(y)$ : the measure  $\mu_y$  satisfies  $\mu = \mu_y \times n$  (under the identification of  $E(y)$  with the fiber of  $Z$  above  $y$ ), and the generalized Radon transform defined by (2.4) can be written

$$Rf(y) = \int_{x \in E(y)} f(x) d\mu_y(x)$$

In a similar way, the measure  $\mu_x$  on each set  $F(x)$  satisfies  $\mu = \mu_x \times m$  (under the identification of the fiber of  $\pi_R$  with  $F(x)$ ), and the dual transform can be written

$$R^*g(x) = \int_{y \in F(x)} g(y) d\mu_x(y)$$

[8] (see also [17, p. 333]).

Since the sets  $F(x)$  are compact, one can compose  $R^*$  and  $R$  for  $f \in C_c(X)$ . We include the uniqueness assumptions  $E(y_1) = E(y_2)$  if and only if  $y_1 = y_2$  and  $F(x_1) = F(x_2)$  if and only if  $x_1 = x_2$ .

Guillemin showed ([5, 6] and with Sternberg [8]) that  $R$  is a Fourier integral distribution associated with integration over  $Z$  and canonical relation  $\mathcal{C} = (N^*(Z) \setminus \{\mathbf{0}\})'$ . To understand the properties of  $R^*R$ , one must investigate the mapping properties of  $\mathcal{C}$ . Let  $\Pi_L : \mathcal{C} \rightarrow T^*(Y)$  and  $\Pi_R : \mathcal{C} \rightarrow T^*(X)$  be the projections. Then we have the following diagram:

$$(2.5) \quad \begin{array}{ccc} & \mathcal{C} & \\ \Pi_L \swarrow & & \searrow \Pi_R \\ T^*(Y) & & T^*(X) \end{array}$$

This diagram is the microlocal version of (2.3).

**DEFINITION 2.1** ([5, 6]). Let  $X$  and  $Y$  be manifolds with  $\dim(Y) = \dim(X)$  and let  $\mathcal{C} \subset (T^*(Y) \times T^*(X)) \setminus \{\mathbf{0}\}$  be a canonical relation. Then,  $\mathcal{C}$  satisfies the *Bolker Assumption* if

$$\Pi_Y : \mathcal{C} \rightarrow T^*(Y)$$

is an injective immersion.

This definition was originally proposed by Guillemin [5],[6, p. 152], [8, p. 364-365] because Ethan Bolker proved  $R^*R$  is injective under a similar assumption for a finite Radon transform. Guillemin proved that if the measures that define the Radon transform are smooth and nowhere zero, and if the Bolker Assumption holds (and  $R$  is defined by a double fibration for which  $\pi_R$  is proper), then  $R^*R$  is an elliptic pseudodifferential operator.

Since we assume  $\dim(Y) = \dim(X)$ , if  $\Pi_Y : \mathcal{C} \rightarrow T^*(Y)$  is an injective immersion, then  $\Pi_Y$  maps to  $T^*(Y) \setminus \{\mathbf{0}\}$  and  $\Pi_X$  is also an immersion [10]. Therefore,  $\Pi_X$  maps to  $T^*(X) \setminus \{\mathbf{0}\}$ . So, under the Bolker Assumption,  $\mathcal{C} \subset (T^*(Y) \setminus \{\mathbf{0}\}) \times (T^*(X) \setminus \{\mathbf{0}\})$  and so  $R$  is a Fourier integral operator according to the definition in [19].

We now put our elliptical transform into this framework.

EXAMPLE 2.2. For our transform  $\mathcal{R}$ , the *incidence relation* is

$$(2.6) \quad Z = \{(s, L, x) \in Y \times D_b \mid x \in E(s, L)\}.$$

The double fibration is

$$(2.7) \quad \begin{array}{ccc} & Z & \\ \pi_L \swarrow & & \searrow \pi_R \\ Y & & D_b \end{array}$$

and both projections are fiber maps. These projections define the sets we integrate over: the ellipse  $E(s, L) = \pi_R(\pi_L^{-1}(\{(s, L)\}))$  and the closed curve in  $Y$

$$F(x) = \pi_L(\pi_R^{-1}(\{x\})) = \{(s, \ell(s, x)) \mid s \in [0, 2\pi]\}$$

where

$$(2.8) \quad \ell(s, x) = |x - \gamma_R(s)| + |x - \gamma_T(s)|.$$

Note that  $\pi_R$  is proper and  $F(x)$  is diffeomorphic to the circle.

One chooses measure  $m = dx$  on  $D_b$  and measure  $n = ds dL$  on  $Y$ . For each  $(s, L) \in Y$  one parameterizes the ellipse  $E(s, L) \cap D_b$  by arc length with coordinate  $t$  so that

$$(2.9) \quad x = x(s, L, t) \in E(s, L)$$

is a smooth function of  $(s, L, t)$ . Then,  $Z$  can be parameterized by  $(s, L, t)$  and this gives the measure we use on  $Z$ ,  $\mu = ds dL dt$ . Since the measure on  $Y$  is  $ds dL$  and  $\mu = (ds dL) dt$ , the measure on the fiber of  $\pi_L$  is  $dt$ . This gives measure  $\mu_{(s, L)} = dt$  which is the arc length measure on the ellipse  $E(s, L)$ .

To find the measure on  $F(x)$  note that the factor,  $w(s, x)$ , giving this measure satisfies

$$(2.10) \quad ds dL dt = w(s, x) ds dx, \quad \text{or} \quad dL dt = w(s, x) dx.$$

For fixed  $s$ ,  $(L, t) \mapsto x(s, L, t)$  give coordinates on  $D_b$ . The Jacobian factor  $w(s, x)$  in equation (2.10) must be

$$(2.11) \quad w(s, x) = |\partial_x \ell| |\partial_x t|$$

where  $L = \ell(s, x)$  and  $t$  are considered as functions of  $x$  and where  $\partial_x$  is the gradient in  $x$  and  $\partial_t$  is the derivative in  $t$ . This expression is valid since the vectors in (2.11) are perpendicular because the first vector is normal to the ellipse  $E(s, L)$  at  $x(s, L, t)$  and the second vector is tangent to the ellipse. Since  $t$  parameterizes arc length, the second factor on the right-hand side of (2.11) is 1. This means  $w(s, x) = |\partial_x \ell(s, x)|$ . To calculate this expression for  $w(s, x)$  we note that

$$\partial_x \ell(s, x) = \frac{x - \gamma_R(s)}{|x - \gamma_R(s)|} + \frac{x - \gamma_T(s)}{|x - \gamma_T(s)|}.$$

Since this expression is the sum of two unit vectors, its length is  $2 \cos(\varphi/2)$  where  $\varphi$  is the angle between these two vectors. A calculation shows that this is

$$(2.12) \quad w(s, x) = 2 \cos(\varphi/2) = \sqrt{2 + 2 \left( \frac{(x - \gamma_R(s)) \cdot (x - \gamma_T(s))}{|x - \gamma_R(s)| |x - \gamma_T(s)|} \right)}$$

where  $\cos \varphi$  is the expression in parentheses in the square root. Note that  $\varphi < \pi$  since  $x$  is not on the segment between the two foci. Therefore, the weight  $w(s, x) \neq 0$ . The second expression (found using the law of cosines) gives  $w$  explicitly in terms of  $s$  and  $x$ .

This discussion shows that  $\mathcal{R}$  and  $\mathcal{R}^*$  satisfy the conditions outlined in the first part of this section so that Guillemin and Sternberg's framework can be applied.

### 3. The Main Result

We now state the main result of this article. Proofs are in Section 5.

**THEOREM 3.1.** *Let  $\alpha \in (0, \pi/2)$  be a constant and let*

$$\begin{aligned} \gamma_T(s) &= (\cos(s - \alpha), \sin(s - \alpha)) \quad \text{and} \\ \gamma_R(s) &= (\cos(s + \alpha), \sin(s + \alpha)) \quad \text{for } s \in [0, 2\pi] \end{aligned}$$

*be the trajectories of the ultrasound emitter and receiver respectively. Denote by  $\mathcal{E}'(D_b)$  the space of distributions supported in the open disc,  $D_b$ , of radius  $b$  centered at 0, where  $b = \cos \alpha$ .*

*The elliptical Radon transform  $\mathcal{R}$  when restricted to the domain  $\mathcal{E}'(D_b)$  is an elliptic Fourier integral operator (FIO) of order  $-1/2$ . Let  $\mathcal{C} \subset T^*(Y) \times T^*(D_b)$  be the canonical relation associated to  $\mathcal{R}$ . Then,  $\mathcal{C}$  satisfies the Bolker Assumption (Definition 2.1).*

As a consequence of this result, we have the following corollary.

**COROLLARY 3.2.** *The composition of  $\mathcal{R}$  with its  $L^2$  adjoint  $\mathcal{R}^*$  when restricted as a transformation from  $\mathcal{E}'(D_b)$  to  $\mathcal{D}'(D_b)$  is an elliptic pseudo-differential operator of order  $(-1)$ .*

This corollary shows that, for  $\text{supp } f \subset D_b$ , the singularities of  $\mathcal{R}^*\mathcal{R}f$  (as a distribution on  $D_b$ ) are at the same locations and co-directions as the singularities of  $f$ , that is, the wavefront sets are the same. In other words,  $\mathcal{R}^*\mathcal{R}$  reconstructs all the singularities of  $f$ . In the next section, we will show reconstructions from an algorithm.

**REMARK 3.3.** Our theorems are valid for any elliptic FIO that has the canonical relation  $\mathcal{C}$  given by (5.1) because the composition calculus of FIO is determined by the canonical relation. This means that our theorems would be true for any other model of this bistatic ultrasound problem having the same canonical relation  $\mathcal{C}$ .

### 4. Reconstructions from a Local Backprojection Algorithm

In this section, we describe a local backprojection type algorithm and show reconstructions from simulated data. The reconstructions and algorithm development were a part of an REU project and senior honors thesis [12] of Tufts University undergraduate Howard Levinson. Prof. Quinto's algorithm

$$(4.1) \quad \mathcal{L}f = \mathcal{R}^*(-\partial^2/\partial L^2)\mathcal{R}f$$

is a generalization of Lambda Tomography [3, 2], which is a filtered backprojection type algorithm with a derivative filter. Note that the algorithm is local in the sense that one needs only data over ellipses near a point  $x \in D_b$  to reconstruct  $\mathcal{L}(f)(x)$ . We infer from our next theorem that  $\mathcal{L}$  detects all singularities inside  $D_b$ .

**THEOREM 4.1.** *The operator  $\mathcal{L} : \mathcal{E}'(D_b) \rightarrow \mathcal{D}'(D_b)$  is an elliptic pseudodifferential operator of order one.*



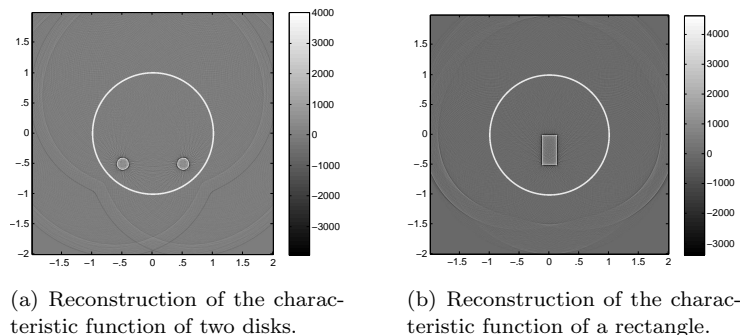


FIGURE 3. Reconstructions using the operator  $\mathcal{L}$  in (4.1) by Tufts Senior Honors Thesis student Howard Levinson [12]. The reconstructions were done with 300 values of  $L$  and 360 values of  $s$ , and  $\alpha = \pi/32$ .

PROOF. The order of  $\mathcal{L}$  is one because  $\mathcal{R}$  and  $\mathcal{R}^*$  are both of order  $-1/2$  and  $-\partial^2/\partial L^2$  is of order two.  $\mathcal{L}$  is elliptic for the following reasons. From Theorem 3.1, we know that  $\mathcal{R}$  is elliptic for distributions in  $\mathcal{E}'(D_b)$ . Then,  $-\partial^2/\partial L^2$  is elliptic on distributions with wavefront in  $\Pi_L(\mathcal{C})$  because the  $dL$  component of such distributions is never zero as can be seen from (5.1). Finally, by the Bolker Assumption on  $\mathcal{C}$ , one can compose  $\mathcal{R}^*$  and  $-\partial^2/\partial L^2\mathcal{R}$  to get an elliptic pseudodifferential operator for distributions supported on  $D_b$ .  $\square$

Mr. Levinson also tried replacing  $-\partial^2/\partial L^2$  by  $-\partial^2/\partial s^2$  in (4.1) but some boundaries were not as well-defined in the reconstructions. This reflects the fact that the analogous operator corresponding to  $\mathcal{L}$  is not elliptic since the symbol of  $-\partial^2/\partial s^2$  is zero on a subset of  $\Pi_L(\mathcal{C})$ . For instance, the  $\mathbf{ds}$  component is 0 on covectors corresponding to points on the minor axis of the ellipse  $E(s, L)$  determined by  $s$  and  $L$ .

REMARK 4.2. These reconstructions are consistent with Theorem 4.1 since all singularities of the objects are visible in the reconstructions and no singularities are added inside  $D_b$ . Notice that there are added singularities in the reconstructions in Figure 3, but they are outside  $D_b$ . Added singularities are to be expected because of the left-right ambiguity: an object on one side of the major axis of an ellipse has the same integral over that ellipse as its mirror image in the major axis. This is most pronounced in the common-offset case in which the foci  $\gamma_T$  and  $\gamma_R$  travel on a line [11].

## 5. Proofs of Theorem 3.1 and Corollary 3.2

Note that if  $g$  is a function of  $(s, x) \in [0, 2\pi] \times D_b$ , then  $\partial_s g$  is the first derivative of  $g$  with respect to  $s$ , and  $\partial_x g$  is the derivative of  $g$  with respect to  $x$ . When  $x = (x_1, x_2) \in \mathbb{R}^2$ , we define  $\partial_x g \mathbf{dx} = \frac{\partial g}{\partial x_1} \mathbf{dx}_1 + \frac{\partial g}{\partial x_2} \mathbf{dx}_2$ . Here we use boldface for covectors, such as  $\mathbf{dx}$ ,  $\mathbf{ds}$ , and  $\mathbf{dL}$  to distinguish them from measures, such as  $dx$ ,  $ds$ , and  $dL$ .

PROOF OF THEOREM 3.1. First, we will calculate  $\mathcal{C} = (N^*(Z) \setminus \{\mathbf{0}\})'$  where  $Z$  is given by (2.6) and then show that  $\mathcal{C}$  satisfies the Bolker Assumption. The set  $Z$  is defined by  $L - \ell(s, x) = 0$  where  $\ell$  is defined by (2.8) and the differential of this function is a basis for  $N^*(Z)$ . Therefore,  $\mathcal{C} = N^*(Z) \setminus \{\mathbf{0}\}'$  is given by

$$(5.1) \quad \mathcal{C} = \{(s, L, -\omega \partial_s \ell \mathbf{d}s + \omega \mathbf{dL}, x, -\omega \partial_x \ell \mathbf{d}x) \mid (s, L) \in Y, x \in E(s, L), \omega \neq 0\}.$$

The Schwartz kernel of  $\mathcal{R}$  is integration on  $Z$  (e.g., [17, Proposition 1.1]) and so  $\mathcal{R}$  is a Fourier integral distribution associated to  $\mathcal{C}$  [6].

We now show that the projection

$$(5.2) \quad \begin{aligned} \Pi_L(s, L, -\omega \partial_s \ell(s, x) \mathbf{d}s + \omega \mathbf{dL}, x, \omega \partial_x \ell(s, x) \mathbf{d}x) \\ = (s, L, -\omega \partial_s \ell(s, x) \mathbf{d}s + \omega \mathbf{dL}) \end{aligned}$$

is an injective immersion. Let  $(s, L, \eta_s, \eta_L)$  be coordinates on  $T^*(Y)$ . Note that  $s, L$  and  $\omega = \eta_L$  are determined by  $\Pi_L$ , so we just need to determine  $x \in D_b$  from (5.2). From the value of  $L$  in (5.2) we know that  $x \in E(s, L) \cap D_b$ , so we fix  $L$ . By rotation invariance, we can assume  $s = 0$ . Now, we let

$$E_b = E(0, L) \cap D_b.$$

Let  $m$  be the length of the curve  $E_b$  and let  $x(t)$  be a parametrization of  $E_b$  by arc length for  $t \in (0, m)$  so that  $x(t)$  moves up  $E_b$  ( $x_2$  increases) as  $t$  increases.

The  $\eta_s$  coordinate in (5.2) with  $x = x(t)$  and  $\omega = -1$  is

$$(5.3) \quad \eta_s(t) = \partial_s \ell(x, s) = \frac{x(t) - \gamma_R(0)}{|x(t) - \gamma_R(0)|} \cdot \gamma'_R(0) + \frac{x(t) - \gamma_T(0)}{|x(t) - \gamma_T(0)|} \cdot \gamma'_T(0).$$

To show  $\Pi_L$  is an injective immersion, we show that  $\eta_s(t)$  has a positive derivative everywhere on  $(0, m)$ . To do this, we consider the terms in (5.3) separately.

The first term

$$(5.4) \quad T_1(t) = \frac{(x(t) - \gamma_R(0))}{|x(t) - \gamma_R(0)|} \cdot \gamma'_R(0)$$

is the cosine of the angle,  $\beta_1(t)$ , between the vector  $(x(t) - \gamma_R(0))$  and the tangent vector  $\gamma'_R(0)$ :

$$T_1(t) = \cos(\beta_1(t)).$$

The vector  $x(t) - \gamma_R(0)$  is transversal to the ellipse  $E(s, L)$  at  $x(t)$  since  $\gamma_R(0)$  is inside the ellipse and  $x(t)$  is on the ellipse. Therefore,  $\beta'_1(t) \neq 0$  for all  $t \in (0, m)$ . Since  $x(t)$  is inside the unit disk,  $\beta_1(t)$  is neither 0 nor  $\pi$  so  $T_1(t) = \cos \beta_1(t)$  is neither maximum or minimum. This implies that  $T'_1(t) \neq 0$  for all  $t \in (0, m)$ . By the Intermediate Value Theorem  $T'_1$  must be either positive or negative everywhere on  $(0, m)$ . Since  $x(t)$  travels up  $E_b$  as  $t$  increases,  $T_1(t) = \cos(\beta_1(t))$  increases, and so  $T'_1(t) > 0$  for all  $t \in (0, m)$ . A similar argument shows that the second term in (5.3) has positive derivative for  $t \in (0, m)$ . Therefore,  $\partial_t \eta_s(t) > 0$  for all  $t \in (0, m)$  and the Inverse Function Theorem shows that the function  $\eta_s(t)$  is invertible by a smooth function. This proves that  $\Pi_L$  is an injective immersion.

As mentioned after Definition 2.1, the projections  $\Pi_L$  and  $\Pi_R$  map away from the  $\mathbf{0}$  section. Therefore,  $\mathcal{R}$  is a Fourier integral operator [19]. Since the measures  $\mu$ ,  $dx$  and  $ds dL$  are nowhere zero,  $\mathcal{R}$  is elliptic. The order of  $\mathcal{R}$  is given by  $(\dim(Y) - \dim(Z))/2$  (see e.g., [6, Theorem 1] which gives the order of  $\mathcal{R}^* \mathcal{R}$ ). In our case,  $Z$  has dimension 3 and  $Y$  has dimension 2, hence  $\mathcal{R}$  has order  $-1/2$ . This concludes the proof of Theorem 3.1.  $\square$

REMARK 5.1. One can extend  $x(t)$  to parametrize the entire ellipse  $E(0, L)$  and consider whether  $\partial_t \eta_s(t) > 0$  for all  $t$  and therefore, whether  $\Pi_L$  is an injective immersion everywhere. However, the proof that  $T_1'(t) > 0$  breaks down when the point  $x$  is one of the two points of intersection of the ellipse  $E(0, L)$  and the tangent line to the unit circle at  $\gamma_R(0)$  because  $T_1'(t) = 0$  at those points. The proof that the second term in (5.3) has positive derivative breaks down at the two points of intersection of  $E(0, L)$  and the tangent line to the unit circle at  $\gamma_T(0)$ . However, the proof can be extended to show that  $\partial_t \eta_s(t) \neq 0$  for both terms in (5.3) for all points  $x(t)$  on  $E(0, L)$  to the left of these two tangent lines.

This does not mean  $\mathcal{R}$  is elliptic inside the unit disk because, for  $L \approx 2a$ , parts of the ellipse  $E(s, L)$  on both sides of the major axis are inside the unit disk and  $\Pi_L$  is not injective. For example, conormals at the two vertices on the minor axis correspond to covectors in  $\mathcal{C}$  mapping to the same covector under  $\Pi_L$ .

PROOF OF COROLLARY 3.2. The proof that  $\mathcal{R}^*\mathcal{R}$  is an elliptic pseudodifferential operator follows from Guillemin's result [6, Theorem 1] as a consequence of Theorem 3.1 and the fact  $\pi_R : Z \rightarrow \mathbb{R}^2$  is proper. We will outline the proof since the proof for our transform is simple and instructive. As discussed previously, we can compose  $\mathcal{R}^*$  and  $\mathcal{R}$  for distributions in  $\mathcal{E}'(D_b)$ .

By Theorem 3.1,  $\mathcal{R}$  is an elliptic Fourier integral operator associated with  $\mathcal{C}$ . By the standard calculus of FIO,  $\mathcal{R}^*$  is an elliptic FIO associated to  $\mathcal{C}^t$ . Because the Bolker Assumption holds above  $D_b$ ,  $\mathcal{C}$  is a local canonical graph and so the composition  $\mathcal{R}^*\mathcal{R}$  is a FIO for functions supported in  $D_b$ . Now, because of the injectivity of  $\Pi_Y$ ,  $\mathcal{C}^t \circ \mathcal{C} \subset \Delta$  where  $\Delta$  is the diagonal in  $(T^*(D_b) \setminus \{\mathbf{0}\})^2$  by the clean composition of Fourier integral operators [1].

To show  $\mathcal{C} = \Delta$ , we need to show  $\Pi_R : \mathcal{C} \rightarrow T^*(D_b) \setminus \{\mathbf{0}\}$  is surjective. This will follow from (5.1) and a geometric argument. Let  $(x, \xi) \in T^*(D_b) \setminus \{\mathbf{0}\}$ . We now prove there is a  $(s, L) \in Y$  such that  $(x, \xi)$  is conormal the ellipse  $E(s, L)$ . First note that any ellipse  $E(s, L)$  that contains  $x$  must have  $L = |x - \gamma_R(s)| + |x - \gamma_T(s)|$ . As  $s$  ranges from 0 to  $2\pi$  the normal line at  $x$  to the ellipse  $E(s, |x - \gamma_R(s)| + |x - \gamma_T(s)|)$  at  $s$  rotates completely around  $2\pi$  radians and therefore for some value of  $s_0 \in [0, 2\pi]$ ,  $(x, \xi)$  must be conormal  $E(s_0, |x - \gamma_R(s_0)| + |x - \gamma_T(s_0)|)$ . Since the ellipse is given by the equation  $L = |x - \gamma_R(s)| + |x - \gamma_T(s)|$ , its gradient is normal to the ellipse at  $x$ ; conormals co-parallel this gradient are exactly of the form  $\xi = \omega \left( \frac{x - \gamma_R(s_0)}{|x - \gamma_R(s_0)|} + \frac{x - \gamma_T(s_0)}{|x - \gamma_T(s_0)|} \right) \mathbf{dx}$  for some  $\omega \neq 0$ . Using (5.1), we see that for this  $s_0$ ,  $x$ ,  $\omega$  and  $L = |x - \gamma_R(s_0)| + |x - \gamma_T(s_0)|$ , there is a  $\lambda \in \mathcal{C}$  with  $\Pi_R(\lambda) = (x, \xi)$ . This finishes the proof that  $\Pi_R$  is surjective. Note that one can also prove this using the fact that  $\pi_R$  is a fibration (and so a submersion) and a proper map, but our proof is elementary. This shows that  $\mathcal{R}^*\mathcal{R}$  is an elliptic pseudodifferential operator viewed as an operator from  $\mathcal{E}'(D_b) \rightarrow \mathcal{D}'(D_b)$ . Because the order of  $\mathcal{R}^*$  and  $\mathcal{R}$  are  $-1/2$ , the order of  $\mathcal{R}^*\mathcal{R}$  is  $-1$ .  $\square$

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